Math. J. Okayama Univ. 27 (1985), 45—47

ON π-REGULAR RINGS WITH INVOLUTION

Dedicated to Professor Hisao Tominaga on his 60th birthday

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Throughout, R will represent a ring with Jacobson radical J. A ring R is called π-regular if for every x in R there exists a positive integer n and a y in R such that $x^n y x^n = x^n$. If R is a ring with involution $*$ and I is an ideal of R which is stable under $*$, then $*$ induces naturally an involution of R/I, which also will be denoted by $*$.

In [2], T. Yanai and the present author classified (von Neumann) regular rings with involution containing no non-trivial symmetric idempotents, and established the structure of regular rings with involution containing only finitely many symmetric idempotents. In this paper, we shall generalize the main results in [2] as follows :

Theorem 1. *Let R be a π-regular ring with involution $*$. Suppose R contains no non-trivial symmetric idempotents. Then R/J is a division ring, the direct sum of a division ring and its opposite, or the ring of 2×2 matrices over a field.*

Theorem 2. *Let R be a π-regular ring with involution $*$. If R contains only finitely many symmetric idempotents, then R/J is isomorphic to a finite direct sum of rings of the following types :*

1. *a division ring,*
2. *the direct sum of a division ring D and its opposite D^{op} with $(a, b)^* = (b, a)$,*
3. *the ring $M_2(K)$ of 2×2 matrices over a field K with $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^* = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$,*
4. *a finite dimensional matrix ring over a finite field,*
5. *the direct sum of a finite dimensional matrix ring $M_n(F)$ over a field F and its opposite $M_n(F)^{op}$ with $(a, b)^* = (b, a)$.*

We begin with the following key proposition.

Proposition 1. *Let R be a π-regular ring with involution $*$, and I a*

**-stable ideal of R. If $u_1, \ldots, u_p$ are orthogonal symmetric idempotents in $\bar{R} = R/I$, then there exist orthogonal symmetric idempotents $e_1, \ldots, e_p$ in R such that $\bar{e}_i = u_i (i = 1, \ldots, p)$.*

Proof. Let u be a symmetric idempotent of $\bar{R}$, and choose an element y in R such that $\bar{y} = u$. Since R is π-regular, there exists a positive integer n and an x in R such that $(yy^*)^{2n}x(yy^*)^{2n} = (yy^*)^{2n}$. Then we have $(yy^*)^{2n}x^*(yy^*)^{2n} = (yy^*)^{2n}$. Now, we set $e = (yy^*)^n x(yy^*)^{2n}x^*(yy^*)^n$. As is easily seen, e is a symmetric idempotent of R. Noting that u is a symmetric idempotent, we have $\bar{y} = \bar{y}^* = u$ and $\bar{e} = (\bar{y}\bar{y}^*)^{2n}\bar{x}^*(\bar{y}\bar{y}^*)^{2n}\bar{x}^*(\bar{y}\bar{y}^*)^{2n} = (\bar{y}\bar{y}^*)^{2n} = u$. This proves the case $p = 1$. We assume $p > 1$ and proceed by induction on p. We may assume that we have found orthogonal symmetric idempotents $e_2, \ldots, e_p$ in R such that $\bar{e}_i = u_i (i = 2, \ldots, p)$. Put $e_0 = e_2 + \ldots + e_p$, which is a symmetric idempotent. Choose an element f in R with $\bar{f} = u_1$. Then $\bar{e}_0\bar{f} = \bar{f}\bar{e}_0 = 0$. For the element $z = (1 - e_0)f(1 - e_0)$ (formally written), there exists a positive integer m and a v in R such that $(zz^*)^{2m}v(zz^*)^{2m} = (zz^*)^{2m}$. Then, setting $e_1 = (zz^*)^m v (zz^*)^{2m} v^*(zz^*)^m$, we see that e_1 is a symmetric idempotent with $\bar{e}_1 = u_1$. Since e_0 is symmetric, we have $z^*e_0 = (e_0 z)^* = 0$. We can now easily see that $e_1 e_0 = e_0 e_1 = 0$. This completes the induction.

Similarly, we can prove the next, whose proof may be left to readers.

Proposition 2. *Let R be a π-regular ring, and I an ideal of R. If $u_1, \ldots, u_p$ are orthogonal idempotents in $\bar{R} = R/I$, then there exist orthogonal idempotents $e_1, \ldots, e_p$ in R such that $\bar{e}_i = u_i (i = 1, \ldots, p)$.*

In [1], Herstein and Montgomery studied rings in which every symmetric element is nilpotent or invertible. By making use of their result, we can prove Theorem 1.

Proof of Theorem 1. In view of [1, Theorem 7], it suffices to show that every symmetric element of R is nilpotent or invertible.

Let y be a non-nilpotent symmetric element of R. Since R is π-regular, there exists a positive integer n and an x in R such that $y^{2n}xy^{2n} = y^{2n}$. Then $e = y^n xy^{2n}x^*y^n$ is a non-zero symmetric idempotent. By hypothesis, this means $e = 1$, and hence y is invertible.

A symmetric idempotent of a ring R with involution is said to be minimal if it cannot be represented as a sum of two non-zero orthogonal symmetric idempotents.

Corollary 1. *Let R be a π-regular ring with involution $*$. If R contains no infinite number of orthogonal symmetric idempotents, then R/J is Artinian.*

Proof. In view of Proposition 1, R/J also contains no infinite number of orthogonal symmetric idempotents. Hence R/J has an identity 1, which is a sum of orthogonal minimal symmetric idempotents. By Theorem 1, every minimal symmetric idempotent of R/J is either a sum of two orthogonal primitive idempotents or itself a primitive idempotent. Hence, 1 is a sum of orthogonal primitive idempotents. This implies that R/J is Artinian.

Proof of Theorem 2. By Corollary 1, R/J is a semisimple Artinian ring. By Proposition 1, every symmetric idempotent of R/J can be lifted to an symmetric idempotent of R, and so R/J has only finitely many symmetric idempotents. Thus, our assertion follows directly from [2, Theorem 2].

REFERENCES

[1] I. N. HERSTEIN and S. MONTGOMERY: Invertible and regular elements in rings with involution, J. Algebra 25 (1973), 390–400.

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(*Received July 15, 1985*)