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ON RINGS IN WHICH ALL COMMUTATORS ARE STRONGLY REGULAR

Dedicated to Professor Akira Hattori on his 60th birthday

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In [3], Herstein proved that if R is an associative ring with the property that for each pair of elements x, y in R there exists an integer $n = n(x,y) > 1$ such that $xy-yx = (xy-yx)^n$ then R is commutative. Putcha, Wilson and Yaqub [6] attempted to weaken the assumption on R, and investigated the structure of a ring R with center Z satisfying the condition that for each pair of elements x, y in R there exists $z = z(x,y) \in Z$ and an integer $n = n(x,y) > 1$ such that $xy-yx = (xy-yx)^n z$. They showed that for such a ring R, R/J is a subdirect sum of division rings and $(xy-yx)^{n-1}$ is in Z, where J denotes the Jacobson radical of R. They claimed also that every generalized quaternion division algebra satisfies the condition. In this paper, we shall show that such a ring R is a subdirect sum of a commutative ring and central division algebras of degree 2, and the condition on R is equivalent to that the commutator ideal of R is a strongly regular ring satisfying the standard polynomial identity S_4 of degree 4. More generally, we shall give some characterizations of a ring in which all commutators are strongly regular, where an element a of a ring R is called strongly regular if $a \in a^2R \cap Ra^2$ (see [1]). Clearly, all commutators in the rings mentioned above are strongly regular. Using a result of Fisher and Snider [2], we shall prove that if all commutators in a ring R are strongly regular then the commutator ideal of R is strongly regular, and R is a subdirect sum of a commutative ring and division rings. Finally, we shall generalize [6, Theorem 5] as follows : if R is a ring with center Z and if for each pair of elements x, y in R there exists an element $z = z(x,y)$ in Z and an even positive integer $n = n(x, y)$ such that $xy-yx = (xy-yx)^n z$, then R is commutative.

Throughout this paper, R denotes an associative ring not necessarily having a unity, $Z(= Z(R))$ the center of R, and $[x,y]$ the commutator $xy-yx$ of x and y in R. The ideal of R generated by all commutators is called the *commutator ideal* of R and is denoted by $C(R)$. An element a of R is called *left π-regular* (resp. *right π-regular*) if there exists an x in R and a positive integer n such that $a^n = xa^{n+1}$ (resp. $a^n = a^{n+1}x$). A left and right π-regular element is called *strongly π-regular*. A ring R is called *left π-regular* if every element of R is left π-regular. In view of a theorem of Dischinger-Zöschinger (see e.g., [5, Proposition 2]), every left π-regular ring is *strongly π-regular*, that is, every element in R is strongly π-regular.

The following lemma has been proved in the proof of [2, Proposition 2.1].

Lemma 1. *An element a of R is left (resp. right) π-regular if and only if so is the natural homomorphic image of a in each prime factor ring of R.*

Proposition 1. *If every prime factor ring of R is commutative or strongly π-regular, then commutator ideal $C(R)$ is strongly π-regular.*

Proof. Let a be an arbitrary element of $C(R)$, and P an arbitrary prime ideal of R. If R/P is commutative, then $\bar{a} = a+P$ equals 0 (and strongly π-regular) in R/P. Hence, by Lemma 1, a is strongly π-regular in R. Now, it is easy to see that a is strongly π-regular in $C(R)$.

For a ring satisfying a polynomial identity, we have

Corollary 1. *If R is a PI-ring, then the following are equivalent:*

(a) *$C(R)$ is strongly π-regular.*

(b) *Every prime factor ring of R is commutative or Artinian simple.*

Proof. It suffices to show that (a) implies (b). Let P be a prime ideal of R, and suppose that R/P is not commutative. Then $I = C(R/P)$ ($\neq 0$) is a strongly π-regular prime PI-ring, and [7, Theorem 1.7.9] proves that I coincides with the ring of central quotients of I, which is an Artinian simple ring with unity e. Let r be an arbitrary element of R/P. Then $I(r-er) = 0$. Since R/P is prime, we have $r = er$. Similarly, $r = re$. Therefore, e is the unity of R. This implies that $R/P = I$, and so R/P is Artinian simple.

The next is [1, Lemma 1].

Lemma 2. *Let a be a strongly regular element of R. Then there exists uniquely an element z in R such that $az = za$, $a^2z = a$ and $az^2 = z$. Moreover, z commutes with every element of R which commutes with a.*

A ring R is called *strongly regular* if all elements of R are left regular, or equivalently, strongly regular. As is well known, a ring R is strongly regular if and only if R is von Neumann regular and every idempotent in R is central. Moreover, in view of Lemma 2, we can easily see that R is strongly regular if and only if R is a strongly π-regular ring without non-zero nilpotent elements.

A ring R is said to be $\cap$-*irreducible* if the intersection of any two non-zero ideals of R is non-zero.

We shall characterize a ring R with $C(R)$ strongly regular.

Theorem 1. *The following are equivalent for a ring R:*

(a) *$C(R)$ is strongly regular.*

(b) *All commutators in R are strongly regular.*

(c) *Every $\cap$-irreducible factor ring of R is a commutative ring or a division ring.*

(d) *R is a subdirect sum of a commutative ring and division rings, and every prime factor ring of R is a commutative ring or a division ring.*

Proof. (a) $\Rightarrow$ (b). This is trivial.

(b) $\Rightarrow$ (c). It suffices to show that if R is a non-commutative $\cap$-irreducible ring satisfying (b) then R is a division ring. First, we claim that every idempotent of R is central. Let e be an idempotent in R and let $a \in R$. Then we have $[e, ea-eae] \in [e, ea-eae]^2R = 0$, that is, $ea = eae$. Similarly, we have $ae = eae$, and so $ea = ae$; e is central. Let x be an arbitrary element of R not contained in Z. Then $[x, y] \neq 0$ for some $y \in R$. By our assumption and Lemma 2, there exists $z \in R$ such that $[x, y] = [x, y]z[x, y]$. Since R is $\cap$-irreducible, the non-zero central idempotent $[x, y]z$ must be the unity of R, and so $[x, y]$ is invertible. Then $x[x, y] = [x, xy]$ implies that x is invertible. Now, let c be an arbitrary non-zero element in Z. Then $c[x, y] = [x, cy]$ implies that c is invertible. Thus we have shown that R is a division ring.

(c) $\Rightarrow$ (d). Noting that every subdirectly irreducible ring and every prime ring are $\cap$-irreducible, we can easily see that (c) implies (d).

(d) $\Rightarrow$ (a). By Proposition 1, $C(R)$ is a strongly π-regular ring. Since R is a subdirect sum of a commutative ring and division rings, we can easily see that $C(R)$ has no non-zero nilpotent elements. Thus, $C(R)$ is strongly regular.

As an immediate corollary to Theorem 1, we have

Corollary 2. *Let R be a ring in which every commutator is strongly* *regular. If there exists no non-commutative division ring which is a homomorphic image of R, then R is commutative.*

In [6], Putcha, Wilson and Yaqub investigated the structure of rings satisfying the following condition:

(I) For every pair of elements x, y in R, there exists an integer $n = n(x,y) > 1$ and an element $z = z(x,y)$ in Z such that $[x,y] = [x,y]^n z$.

By making use of Theorem 1, we shall characterize a ring satisfying (I).

Theorem 2. *The following are equivalent for a ring R:*

(a) *R satisfies* (I).

(b) *$C(R)$ is a strongly regular ring satisfying the standard polynomial identity S_4 of degree* 4.

(c) *Every $\cap$-irreducible factor ring of R is a commutative ring or a central division algebra of degree* 2.

(d) *R is a subdirect sum of a commutative ring and central division algebras of degree* 2, *and every prime factor ring of R is a commutative ring or a division ring.*

Proof. Clearly, (c) implies (d), and Theorem 1 shows that (d) implies (b).

(a) $\Rightarrow$ (c). In view of Theorem 1, it suffices to show that every non-commutative division ring R satisfying (I) is a central division algebra of degree 2. Let x, y be two elements of R. By (I), there exists an integer $n > 1$ and $z \in Z$ such that $[x,y] = [x,y]^n z$. If $[x,y] \neq 0$, then $[x,y]^{n-1} = z^{-1} \in Z$. On the other hand, if $[x,y] = 0$ then $[x,y] \in Z$ trivially. Therefore, by [4, Corollary 3.7], D is a central divison algebra of degree 2.

(b) $\Rightarrow$ (a). By Theorem 1, R is a subdirect sum of a commutative ring and division rings D_λ $(\lambda \in \Lambda)$. Since each D_λ is a homomorphic image of $C(R)$, D_λ satisfies S_4. Thus, each D_λ is a central division algebra of degree 2 by [6, Theorem 1.5.16]. Let D be one of the D_λ and let K be a maximal subfield of D. Then, regarding D as a subring of $D \otimes K = M_2(K)$, by Cayley-Hamilton theorem we see that $[x,y]^2 = \mathrm{tr}([x,y])[x,y] - \det([x,y]) = -\det([x,y]) \in Z(D)$ $(x, y \in D)$. Since R is a subdirect sum of a commutative ring and the D_λ, we have $[x,y]^2 \in Z$ for all x, y in R.

Now, let x, y be arbitrary elements of R, and put $a = [x,y]$. By Lemma 2, there exists uniquely an element $z \in C(R)$ such that $az = za$, $a^2 z = a$ and $az^2 = z$. Then $a^4 z^2 = a^2$ and $a^2 z^4 = z^2$. Since $a^2 \in Z$, we conclude $z^2 \in Z$ by Lemma 2. Also, we can easily see that $a = a^3 z$. Hence,

R satisfies (I).

Corollary 3. *If R contains no infinite set of orthogonal central idempotents, then the following are equivalent*:

(a) *R satisfies* (I).

(b) *R is a direct sum of a commutative ring and a finite number of central division algebras of degree* 2.

Proof. It suffices to show that (a) implies (b). By Theorem 2, every idempotent of R is central. Hence, by hypothesis, $C(R)$ is a finite direct sum of central division algebras of degree 2 (Theorem 2), and $C(R)$ is a direct summand of R.

The following example shows that every ring satisfying (I) need not be a direct sum of a commutative ring and a strongly regular ring.

Example. Let H^N be the direct product of copies of the ring H of real quaternions indexed by the set N of natural numbers, and $H^{(N)}$ the direct sum of copies of H. Consider the subring $R = Z\cdot 1 + H^{(N)}$ of H^N generated by 1 and $H^{(N)}$. Then, $C(R) = H^{(N)}$ is strongly regular, but R cannot be a direct sum of a commutative ring and a strongly regular ring.

Finally, we consider the following condition:

(II) For each pair of elements x, y in R, there exists an element $z = z(x,y)$ in Z and an even positive integer $n = n(x,y)$ such that $[x,y] = [x,y]^n z$.

We conclude this paper with the following corollary which generalizes [6, Theorem 5].

Corollary 4. *Every ring R satisfying* (II) *is commutative*.

Proof. Let x, $y \in R$. As was shown in the proof of Theorem 2, $[x,y]^2 \in Z$, and therefore, by hypothesis, $[x,y] \in Z$. Since any division ring with this property is commutative, Theorem 2 proves that R is commutative.

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