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RINGS DECOMPOSED INTO DIRECT SUMS OF NIL RINGS AND CERTAIN REDUCED RINGS

Dedicated to Professor Noboru Ito on his 60th birthday

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Recently, in his paper [5], M. Ôhori introduced the concept of generalized right (resp. left) p.p. rings with identity. This concept can be extended to rings without identity as follows: An element x of a ring R is called a *right* (resp. *left*) *p. p. element* if there exists an idempotent e in R such that $xe = x$ and $r(x) = r(e)$ (resp. $ex = x$ and $l(x) = l(e)$), where $r(*)$ (resp. $l(*)$) denotes the right (resp. left) annihilator of $*$ in R. A ring R is called a *generalized right* (resp. *left*) *p. p. ring* if for every $x \in R$ there exists a positive integer n such that x^n is a right (resp. left) p. p. element, and R is a right (resp. left) p. p. ring if every $x \in R$ is a right (resp. left) p. p. element.

Obviously, every π-regular ring is a generalized (right and left) p. p. ring, and every direct sum of generalized right (resp. left) p.p. rings whose idempotents are central is also a generalized right (resp. left) p. p. ring. For instance, every (probably infinite) direct sum of domains with identity is a (right and left) p. p. ring.

Throughout the present paper, R will represent a ring. Let N be the set of nilpotent elements in R, and P the set of right p. p. elements in R. Given an integer $q > 1$, we set $E_q = \{x \in R \mid x^q = x\}$; in particular, $E = E_2$.

We consider the following conditions:

(#) Each $x \in R$ has at most one representation of the form $x = x' + x''$, where $x' \in N$ and $x'' \in P$.

(#)' Each $x \in R$ has at most one representation of the form $x = x' + x''$, where $x' \in N$ and x'' is right regular ($x'' = x''^2 y$ for some $y \in R$).

(#)" Each $x \in R$ has at most one representation of the form $x = x' + x''$, where $x' \in N$ and x'' is potent ($x'' = x''^k$ for some integer $k > 1$).

($*$) E is contained in some reduced ideal A of R.

The purpose of this paper is to prove the following theorem which deduces numerous decomposition theorems, among others, [1, Theorem 3] and [6, Theorem 1].

Theorem 1. *The following conditions are equivalent:*

1) *R is a generalized right p. p. ring and satisfies (#).*

2) *R is a generalized right p. p. ring and satisfies (*).*

3) *$R = N \oplus P$; strictly speaking, both N and P are ideals of R and R is the direct sum of N and P.*

When this is the case, P is a reduced (right and left) p. p. ring.

Proof. Obviously, 3) implies 1).

1) $\Rightarrow$ 2). Let e be an arbitrary element of E. Given $x \in R$, we set $u = ex - exe$. Since $u^2 = 0$ and $e+u$ is also in E, (#) implies that $u = 0$, i.e., $ex = exe$. Similarly, we can show that $xe = exe$, and therefore e must be central. Now, let $v \in N \cap eR$. Since $v+e$ is invertible in the ring eR, it is a right p. p. element of R; (#) implies that $v = 0$. Hence eR is a reduced ring. Now, let A be the ideal of R generated by E. Since E is contained in the center of R, for any $a \in A$ there is some $f \in E$ such that $a \in fR$. Therefore, a cannot be a non-zero nilpotent element. Consequently, A is a reduced ideal.

2) $\Rightarrow$ 3). As above, we can easily show that every idempotent of R is central. Furthermore, there holds $A = P$. Let x be an arbitrary element of R. Then, by hypothesis, there exists a positive integer m and an idempotent e such that $x^m e = x^m$ and $r(x^m) = r(e)$. Clearly, $(xe)e = xe$ and $r(xe) = r(e)$, and so $xe \in P$. Since e is central, we have $(x-xe)^m = 0$. Therefore, x is the sum of $x-xe \in N$ and $xe \in P$. Next, we claim that if $u \in N$ and $y \in R$ then yu and $uy \in N$. Actually, there exists a positive integer n and $f \in E$ such that $(yu)^n f = (yu)^n$ and $r((yu)^n) = r(f)$. Let k be the least positive integer such that $u^k f = 0$. If $k > 1$, then $(yu)^n u^{k-1} = (yu)^n fu^{k-1} = (yu)^{n-1} yu^k f = 0$, which forces a contradiction $fu^{k-1} = 0$. We conclude therefore that $(yu)^n = (yu)^n f = 0$ and $(uy)^{n+1} = 0$. In particular, we get $PN = 0 = NP$. Now, let v, $v' \in N$, and $v+v' = w+p$, where $w \in N$ with $w^l = 0$ and $p \in P$. In view of $NP = 0$, we get $(v+v')^2 = (v+v')(w+p) = (v+v')w$, and hence $(v+v')^{l+1} = (v+v')w^l = 0$. Thus we have shown that N forms an ideal of R and $R = N \oplus P$.

Corollary 1. *The following conditions are equivalent:*

1) *R is a π-regular ring and satisfies (#).*

1)′ *R is a π-regular ring and satisfies (#)′.*

2) *R is a π-regular ring and satisfies* ($*$).
3) *$R = N \oplus P$, and P is a strongly regular ring.*

Proof. By Theorem 1 (and its proof), it suffices to show that 1)' implies 2). Since we can show that every idempotent of R is central and R is strongly π-regular, the proof proceeds in the same way as in that of 1) $\Rightarrow$ 2) of Theorem 1.

Now, let R be a P'_n-ring in the sense of [2], that is, $xR^n = xR^n x$ for all $x \in R$. Then $xR^n = xR^n x^k$ for $k = 1, 2, \cdots$; in particular, R is a left π-regular ring with $N^{n+1} = 0$. Hence, by a result of Zöschinger-Dischinger (see, e.g., [3, Proposition 2]), R is strongly π-regular. Furthermore, if $e \in E$ and $u^2 = 0$ then $ue \in uR^n u^2 = 0$ and $eu = e \cdot e^{n-1} u \in eR^n e$, whence we see that $eu = eue = 0 = ue$. This enables us to see that e is central. For any $x \in R$, we now have $ex = exe^n \in exR^n = exR^n(ex)^{n+1}$, which proves that eR is a reduced ideal of R. Hence, R satisfies ($*$). This fact together with Corollary 1 gives the following which includes the main part of [2, Theorem 2].

Corollary 2. *The following conditions are equivalent:*
1) *R is a π-regular ring with $N^{n+1} = 0$ and satisfies* ($\#$).
1)' *R is a π-regular ring with $N^{n+1} = 0$ and satisfies* ($\#$)'.
2) *R is a π-regular ring with $N^{n+1} = 0$ and satisfies* ($*$).
3) *$R = N \oplus P$, P is strongly regular, and $N^{n+1} = 0$.*
4) *R is a P'_n-ring.*

In the same way as for Corollary 1, we can prove the following which includes [1, Theorem 3] and [6, Theorem 1].

Corollary 3. *The following conditions are equivalent:*
1) *R is a periodic ring and satisfies* ($\#$).
1)'' *R is a periodic ring and satisfies* ($\#$)''.
2) *R is a periodic ring and satisfies* ($*$).
3) *$R = N \oplus P$, and P is a J-ring (every element of P is potent).*

Finally, we shall prove the following

Corollary 4. *If R is a ring with 1, then the following conditions are equivalent:*
1) *The addition "+" of R is equationally definable in terms of the*

multiplication "·" and the successor operation "$\wedge$" of R, and R satisfies ($\sharp$) (*or* ($\sharp$)$''$).

2) *There exists a positive integer n such that $R = E_{n+1}$.*

Proof. If R satisfies 1), then there exists a positive integer n such that $x^n = x^{2n}$ for all $x \in R$, by [4, Theorem 1]. Obviously, R is of bounded index at most n. Since $(x^{n+1})^{n+1} = x^{n+1}$ and $(x - x^{n+1})^n = 0$, x is the sum of $x - x^{n+1} \in N$ and $x^{n+1} \in E_{n+1}$. Hence $R = N \oplus E_{n+1}$ by Corollary 3, so that $R = E_{n+1}$. The converse is also clear by [4, Theorem 1].

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