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A CHARACTERIZATION OF BOOLEAN RINGS (III)

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Throughout, R will represent a ring. Let $N = N(R)$ be the set of nilpotent elements in R, $E = E(R)$ the set of idempotents in R, and $E^* = E^*(R)$ the set of central idempotents in R. A ring R is called an *I-ring* (resp. *I'-ring*) if every element of R can be represented as a product of elements in E (resp. $E \cup N$).

In this very brief note, we give the following characterization of Boolean rings.

Theorem 1. *Let R be a ring. Then the following are equivalent:*

1) *R is a Boolean ring.*

2) *For every non-zero $x \in R$, there exists an I'-subring S with $x \in S$ and $E^*(S) \neq 0$ and satisfying the minimum condition on right (or left) annihilators.*

3) *For every non-zero $x \in R$, there exists an I'-subring S with $x \in S$ and $E^*(S) \neq 0$ and satisfying the maximum condition on principal right (or left) ideals.*

Actually, our theorem is a direct consequence of [2, Theorem 1] and the next lemma.

Lemma 1. *If R is an I'-ring and $E^* \neq 0$, then R is an I-ring. In particular, every I'-ring with unity is a Boolean ring.*

Proof. Let e be a non-zero central idempotent in R. First, we claim that $eN = 0$. Obviously, $S = eR$ is an I'-ring with unity e. Let a be any element in N. Then $e - ea = x_1x_2\cdots x_n$ with some $x_i \in eE \cup eN$. Since $e - ea$ is invertible in S, x_1 cannot be nilpotent, so that $x_1 \in eE$. Hence $e - ea = x_1(e - ea)$, and therefore $x_1 = e$, namely $e - ea = x_2\cdots x_n$. Repeating this procedure, we eventually obtain $e - ea = e$, namely $ea = 0$. This proves that $eN = 0$.

Suppose, to the contrary, that there exists an element r of R which cannot be represented as a product of idempotents. Let $e + r = y_1y_2\cdots y_k$ with some $y_i \in E \cup N$. Since $eN = 0$ by the above claim, we see that $e = e(e+r) = (ey_1)(ey_2)\cdots(ey_k)$ and every y_i is an idempotent, and hence $ey_i = e$. This proves that every $y_i - e$ is an idempotent and $r = y_1y_2\cdots y_k - e$

$= (y_1 - e)(y_2 - e)\cdots(y_k - e)$. But, this is impossible. The latter assertion is clear by [2, Lemma 1 (2)].

Corollary 1. *Let R be an I'-ring. If R is a π-regular PI-ring, then N coincides with the prime radical of R and R/N is Boolean.*

Proof. We shall prove the equivalent statement that if R is a π-regular semiprime *PI*-ring then R is Boolean. To see this, it suffices to prove the case that R is prime. Noting that the center of R is non-zero (see, e.g. [1, Theorem 1.4.2]), we can easily see that R has a unity; hence R is a Boolean ring, by Lemma 1.

REFERENCES

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