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ON A THEOREM OF POSNER

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Throughout, R will represent a (differential) ring with the non-zero derivation $d : r \mapsto r'$, $K = \{r \in R \mid r' = 0\}$, and U a non-zero differential ideal of R. Let $\delta : r \mapsto r^*$ be another derivation of R. Given a subset X of R, we set $C_R(X) = \{r \in R \mid rx = xr \text{ for all } x \in X\}$; in particular, $C = C_R(R)$, the center of R. As for definitions and fundamental results used in this paper without mention, we refer to [4, § 3].

The main theorem of this paper is the following generalization of a theorem of Posner [6, Theorem 1].

Theorem 1. *If R is a d-prime ring of characteristic not 2 and $d\delta$ (or δd) induces a derivation of U into R then $\delta = 0$. In particular, if R is a prime ring of characteristic not 2 and $d\delta$ (or δd) induces a derivation of U into R then $\delta = 0$.*

As corollaries to this theorem, we shall give several results concerning prime rings of characteristic not 2 and 2-torsion free d-semiprime rings. In advance of proving Theorem 1, we state the following

Lemma 1. (1) *Let R be a d-prime ring. If $a \in R$ and $aU' = 0$ (or $U'a = 0$) then $a = 0$.*

(2) *Let R be a d-prime ring. If $U^* = 0$ then $\delta = 0$; in particular, $U' \neq 0$.*

(3) *Let P be a d-prime ideal of R with $U' \not\subseteq P$. If $d\delta$ (or δd) induces a derivation of U into R, then $P^* \subseteq P$.*

Proof. (1) As is easily seen, $aUr^{(k)} = 0$ for all $r \in R$ and all positive integers k. Hence, by $R' \neq 0$, we get $a = 0$.

(2) For any $r \in R$ and $u \in U$, we have $r^*u = (ru)^* - ru^* = 0$, i.e., $R^*U = 0$. Hence $R^* = 0$.

(3) Since $d\delta$ (or δd) induces a derivation of U into R, we have

$$(*) \qquad u'v^* + u^*v' = 0 \ (u, v \in U).$$

If $u, v \in U$ and $x \in P$, then $(*)$ shows that

$$ux^*v' = (ux)^*v' - u^*xv' = -(ux)'v^* - u^*xv' \in P.$$

Hence $UP^*U' \subseteq P$. Noting that $U \not\subseteq P$, we get $P^*U' \subseteq P$. Since $U' \not\subseteq P$, (1) proves that $P^* \subseteq P$.

Proof of Theorem 1. By repeated use of $(*)$, we see that for any $u, v, w \in U$,

$$\begin{aligned} 2u^*v'w' &= -u'v^*w' + u^*v'w' \\ &= u'v'w^* + u^*v'w' + u\{(v')'w^* + (v')^*w'\} \\ &= (uv')'w^* + (uv')^*w' = 0. \end{aligned}$$

Hence $U^*U'U' = 0$. Then $U^* = 0$ by Lemma 1 (1), and so $\delta = 0$ by Lemma 1 (2).

Corollary 1. *Let R be a d-prime ring of characteristic not* 2, *and let a be an element of R. If $[a, U'] = 0$, then a is in C.*

Corollary 1 will suggest the conjecture that if a and b are elements of a prime ring R of characteristic not 2 such that $au' = u'b$ for all $u \in U$ then a or b is in C. However, the next example shows that the conjecture is false : Let R be the Hamiltonian quaternion algebra over $\mathbf{R}$: $R = \mathbf{R}\cdot 1 \oplus \mathbf{R}\cdot i \oplus \mathbf{R}\cdot j \oplus \mathbf{R}\cdot ij$, and consider the non-zero inner derivation $d : r \to r' = [i, r]$. Then $ir' - r'(-i) = 0$ for all $r \in R$.

We shall prove the following

Corollary 2. *Let a, b be elements of a prime ring R of characteristic not* 2. *Then the following are equivalent :*

1) $au' = u'b$ *for all* $u \in U$.

2) *Either* $a = b \in C$ *or* $C_R(a) = K$ *and* $a+b, ab \in C$.

Proof. 1) ⇒ 2). If either a or b is in $C_R(U')$, then 1) gives $U'(b-a) = 0$ or $(b-a)U' = 0$, which implies $a = b$ by Lemma 1 (1). Hence $a = b \in C$ by Corollary 1. We assume henceforth that neither a nor b belongs to $C_R(U')$. For any $u, v \in U$, 1) gives $a(uv)' = (uv)'b$, that is, $u'[b, v] = [u, a]v'$. Furthermore, for any $r \in R$ this gives

$$(**) \qquad r'u[b, v] = [r, a]uv'.$$

Now, suppose $r' = 0$. Then $(**)$ shows that $[r, a]uv' = 0$, i.e., $[r, a]UU' = 0$. Hence $[r, a] = 0$ by Lemma 1 (1). Conversely, if $[r, a] = 0$ then $(**)$ shows that $r'U[b, v] = 0$ for any $v \in U$. Since $b \notin C_R(U)$, we get $r' = 0$. Hence $C_R(a) = K$, and similarly $C_R(b) = K$. In particular, $a' =$

$b' = 0$ and $ab = ba$. Then, by 1), for any $u \in U$ we have $au - ub \in K$, and so $a[a, u] - [a, u]b = [a, au - ub] = 0$. By repeated use of this relation, we see that

$$\begin{aligned}[u, a][a+b, v] &= [a, u][v, a] + [u, a][b, v] \\ &= \{au[v, a] + [u, a]bv\} - \{ua[v, a] + [u, a]vb\} \\ &= \{au[v, a] + a[u, a]v\} - \{u[v, a]b + [u, a]vb\} \\ &= a[uv, a] - [uv, a]b = 0 \quad (u, v \in U),\end{aligned}$$

which implies that $[U, a][a+b, v] = 0$ for all $v \in U$. Hence, by Lemma 1 (1), we get $a+b \in C$. Furthermore, $[ab, u] = a[b, u] + [a, u]b = a[b, u] + a[a, u] = a[a+b, u] = 0$ for all $u \in U$, and therefore Corollary 1 proves that ab is in C.

2) $\Rightarrow$ 1). It suffices to consider the case that $C_R(a) = K$ and $a+b$, $ab \in C$. Since $C_R(b) = C_R(a) = K$, we get $(au - ub)' = au' - u'b$ $(u \in U)$. Further, noting that $ab = ba$, we see that $[a, au - ub] = a[a, u] - [a, u]b = a[(a+b) - b, u] - [a, u]b = a[u, b] + [u, a]b = [u, ab] = 0$ $(u \in U)$. Hence $au - ub \in K$, so that $au' = u'b$ for all $u \in U$.

Corollary 3. *Let R be a prime ring. If δ is non-zero and $d\delta$ is also a derivation of R, then R is of characteristic 2 and $d = \alpha\delta$ for some non-zero element α in the extended centroid of R (and conversely).*

Proof. By hypothesis, there exists an element a in R such that $a^* \neq 0$. In view of (*), we can easily see that

$$(\#) \qquad a'rx^* + a^*rx' = a'(rx)^* + a^*(rx)' = 0 \quad (r, x \in R).$$

By [3, Lemmas 1.3.1 and 1.3.2], the extended centroid of R is a field and $a' = -\alpha a^*$ with some α in the extended centroid of R. Then (#) shows that $a^*R(-\alpha x^* + x') = 0$ for all $x \in R$, which implies that $x' = \alpha x^*$ for all $x \in R$. Nothing to say, R is of characteristic 2, by Theorem 1.

Next, we consider the case that R is a 2-torsion free d-semiprime ring, for which Theorem 1 is no longer valid. However, we can prove the following which together with Lemma 1 (2) derives Theorem 1.

Theorem 2. *Let R be a 2-torsion free d-semiprime ring. If $d\delta$ (or δd) induces a derivation of U into R, then there exists a differential ideal V and a δ-stable differential ideal T of R such that $U' \subseteq V$, $R^* \subseteq T$ and $V \cap T = 0$.*

Proof. Let $\{P_\lambda\}_{\lambda\in\Lambda}$ be the set of all d-prime ideals of R, and put $\Lambda_1 = \{\lambda \in \Lambda \mid 2R \not\subseteq P_\lambda\}$ and $\Lambda_2 = \{\lambda \in \Lambda_1 \mid U' \not\subseteq P_\lambda\}$. First, we claim that $\bigcap_{\lambda\in\Lambda_1} P_\lambda = 0$. If not, there exists some $r \in (\bigcap_{\lambda\in\Lambda_1} P_\lambda)\backslash(\bigcap_{\lambda\in\Lambda\backslash\Lambda_1} P_\lambda)$, and then $2r \in \bigcap_{\lambda\in\Lambda} P_\lambda = 0$. But, this contradicts that R is 2-torsion free. Let $\lambda \in \Lambda_2$. Then $P_\lambda^* \subseteq P_\lambda$ by Lemma 1 (3), and hence by Theorem 1 we have $R^* \subseteq P_\lambda$. Therefore if we put $V = \bigcap_{\lambda\in\Lambda_1\backslash\Lambda_2} P_\lambda$ and $T = \bigcap_{\lambda\in\Lambda_2} P_\lambda$, then V and T satisfy the conditions required.

Corollary 4. *Let R be a 2-torsion free d-semiprime ring. If d^2 induces a derivation of U into R, then $U' = 0$. In particular, if $l(U) = 0$ then d^2 cannot induce a derivation of U into R.*

Proof. If d^2 induces a derivation of U into R, then Theorem 2 shows that $U' = 0$. Since $r'u = (ru)' - ru' = 0$ for all $r \in R$ and $u \in U$, we get $R'U = 0$. Hence, if $l(U) = 0$ then $R' = 0$, a contradiction.

Corollary 5. *Let R be a 2-torsion free d-semiprime ring. If U is δ-stable and $d\delta$ (or δd) induces a derivation of U into R, then $(U^*)' = 0$.*

Proof. Under the notation in Theorem 2, we have $(U^*)' \subseteq V \cap T = 0$.

Corollary 6. *Let R be a 2-torsion free semiprime ring. If $d\delta$ (or δd) is a derivation of R then $d\delta = \delta d = 0$.*

A d-prime ideal P of R is said to be *δ-d-prime* if P is d-prime with respect to the derivation δ, and an ideal Q is said to be *δ-d-semiprime* if Q is the intersection of δ-d-prime ideals containing Q. The ring R is called a *δ-d-prime* (resp. *δ-d-semiprime*) ring if 0 is a δ-d-prime (resp. δ-d-semiprime) ideal. As an easy combination of Corollary 1 and [4, Lemma 7], we see that if R is a δ-d-prime ring of characteristic not 2 with non-zero δ, U is δ-stable and $[U', U^*] = 0$, then R is commutative.

Finally, we shall prove the following which may be regarded as a generalization of [4, Theorem 3].

Theorem 3. *Let R be a 2-torsion free δ-d-semiprime ring, and U a δ-stable differential ideal of R with $l(U) = 0$. Let $H = \{r \in R \mid (r')^* = (r^*)' = 0\}$. If H is commutative and $[U', U^*] = 0$, then R is commutative.*

Proof. As is easily seen, $\bigcap_{\gamma \in \Gamma} P_\gamma = 0$ with δ-d-prime ideals P_γ such that $U \not\subseteq P_\gamma$ and R/P_γ is of characteristic not 2 (see the proof of Theorem 2). Put $\Gamma_1 = \{\gamma \in \Gamma \mid R' \not\subseteq P_\gamma \text{ and } R^* \not\subseteq P_\gamma\}$, and let D be the commutator ideal of R. (Note that D is a δ-stable differential ideal.) Then, as was claimed above, $D \subseteq P_\gamma$ for all $\gamma \in \Gamma_1$. If $\gamma \in \Gamma \backslash \Gamma_1$ then either $D' \subseteq P_\gamma$ or $D^* \subseteq P_\gamma$. Thus, $(D')^* \cup (D^*)' \subseteq \bigcap_{\gamma \in \Gamma} P_\gamma = 0$, which implies that $D \subseteq H$. Then, by hypothesis, D is a commutative ideal. (It is easy to see that every d-prime ring containing a commutative non-zero differential ideal is commutative.) Now, the argument employed in the last part of the proof of [3, Theorem 3] enables us to see that $D \subseteq \bigcap_{\gamma \in \Gamma} P_\gamma = 0$, namely R is commutative.

Remark. In [4, § 3], the hypothesis $2R = R$ may be replaced by the weaker one that R is 2-torsion free.

Appendix. In what follows, R will always represent a prime ring of characteristic not 2 with center C and with a non-zero derivation $d : r \mapsto r'$. Let U be a Lie ideal of R, and put $W = [U, U]$ and $S = [W, W]$. Obviously, W and S are Lie ideals of R and $S'' \subseteq W' \subseteq U$. We consider the following conditions :

(1-U) $U \subseteq C$.
(2-U) $U' \subseteq C$.
(3-U) $U'' \subseteq C$.
(4-U) $[U', U'] \subseteq C$.
(5-U) There exists some $a \in R \backslash C$ such that $[a, U'] \subseteq C$.
(6-U) There exists a non-zero $a \in R$ such that $aU' \subseteq C$.
(7-U) $[u, u'] \in C$ for all $u \in U$.
(8-U) There exists a non-zero derivation $\delta : r \mapsto r^*$ such that $(U')^* \subseteq C$.

Needless to say, if R is not commutative, (1-U) implies (2-U)—(8-U). In connection with Corollary 1, we shall reprove all the results in [1] and [5] by giving the following theorem.

Theorem A. *If R is not commutative, then the conditions* (1-U), (2-S), (3-S), (4-S), (5-S), (6-S), (7-S) *and* (8-S) *are equivalent.*

All the preliminary results are summarized in the next lemma.

Lemma B. (1) $(1\text{-}U)$, $(1\text{-}S)$ *and* $(2\text{-}S)$ *are equivalent.*
(2) *If* $U'' = 0$ *then* $U \subseteq C$.
(3) *If* $[a, U'] = 0$ *for some* $a \in R\backslash C$ *then* $U \subseteq C$.
(4) *If* $aU' = 0$ *for some non-zero* $a \in R$ *then* $U \subseteq C$.
(5) *If* $U \not\subseteq C$ *and* $U''' = 0$ *then* $d^3 = 0$.

Proof. (1) is clear by [2, Lemmas 3 and 6], and (2), (3), (4) and (5) are Theorem 1, Theorem 2, Lemma 7 and Lemma 11 in [2], respectively.

Proof of Theorem A. In view of Lemma B(1), it suffices to show that each of $(3\text{-}S)$ and $(4\text{-}U)-(8\text{-}U)$ implies $(1\text{-}U)$.

$(3\text{-}S) \Rightarrow (1\text{-}U)$. Noting that $2[s', t'] = [s, t]'' \in C$ $(s, t \in S)$, we get $(4\text{-}S)$. Suppose, to the contrary, that $S' \not\subseteq C$ (Lemma B(1)). Then $[a, S'] \subseteq C$ for some $a \notin C$. If there exists a $c \in C$ such that $c' \neq 0$, then $c'[a, [s, r]] = [a, [s, cr]']-c[a, [s, r]'] \in C$ $(s \in S,\ r \in R)$, and so $[a, [a, [S, R]]] = 0$. By Lemma B(2), this implies $[S, R] \subseteq C$; in particular, $[a, [a, S]] = 0$. This forces a contradiction $U \subseteq C$ (Lemma B(1) and (2)). Thus, we assume henceforth that $C' = 0$. Since $S''' \subseteq C' = 0$, Lemma B(5) proves that $R''' = 0$. Now, let s be an arbitrary element of S with $s'' = 0$. Then, for any $t \in S$, $[s', t'] \in C$ and $2s'[s', t'] = [s, st']''$ $\in C$. Hence $[s', S'] = 0$, and $s' \in C$ (Lemma B(3)). In what follows, let s be an arbitrary element of S with $s'' \neq 0$. Since $[s, R'']'' = 0$, by what we have just shown above we see that $[s', R''] = [s, R'']' \subseteq C$; in particular, $[s', [s', R'']] = 0$. Thus $s''[s', [s', R']] = [s', [s', (sR')'']] = 0$, whence $[s', [s', R']] = 0$ follows, and therefore $s''[s', [s', R]] = [s', [s', (s'R)']]$ $= 0$. Hence $[s', [s', R]] = 0$, and so $s' \in C$ (Lemma B(2)). We have thus seen that $S' \subseteq C$, which is a contradiction.

Claim. In particular, we have seen that if $[a, [a, S]] \subseteq C$ for some $a \in R\backslash C$ then $U \subseteq C$.

$(5\text{-}U) \Rightarrow (3\text{-}S)$. If there exists a $c \in C$ such that $c' \neq 0$ then, as shown at the beginning of the above proof, $U \subseteq C$. Thus, in what follows, we assume that $C' = 0$. Suppose, to the contrary, that $S'' \not\subseteq C$. For any $u \in U$, $[a', [a', u]] = [a, [a, u]']'-[a, [a', u]'] \in C$. Hence a' is central (Claim), and $2a'[a, [a, u]] = [a, [a^2, u]'] \in C$, so that $a'[a, [a, U]] \subseteq C$. Since $[a, [a, U]] \subseteq C$ (Claim), this implies $a' = 0$. By making use of $[a, [U', U']] = 0$ and $[a, U''] = [a, U']' = 0$, we see that $[a, [w'', u]]$ $= [a, [w', u]'] \in C$ and $a[a, [w'', u]] = [a, [w', au]']-[a, [w', au']] =$

$[a, [w', au]'] - [w', a][a, u'] \in C(u \in U, w \in W)$, and so $[a, [W'', U]] = 0$. But, this gives a contradiction $U \subseteq C$ (Lemma B (3)).

(4-U) $\Rightarrow$ (1-U). If not, (5-U) holds by Lemma B (1). But, we have seen that (5-U) is equivalent to (1-U), a contradiction.

(6-U) $\Rightarrow$ (3-S). If $a \in C$ then $U \subseteq C$. We assume henceforth that $a \notin C$. If there exists a $c \in C$ such that $c' \neq 0$, then $c'a[u, r] = a[u, cr]' - ca[u, r]' \in C(u \in U, r \in R)$, and so $a[a, [U, R]] = [a, a[U, R]] = 0$. This implies $[U, R] \subseteq C$ (Lemma B (4)), and hence $U \subseteq C$ (Lemma B (1)). Thus, in what follows, we assume that $C' = 0$. Suppose, to the contrary, that $S'' \nsubseteq C$. Obviously, $a[a, U'] = [a, aU'] = 0$, $a[a', U] = a[a, U]' \subseteq C$, and $a'w' = (aw')' - aw'' \in C(w \in W)$. Since $a[a', w']w' = a[a'w', w'] = 0$, we have $a[a', w'] = 0$, namely $a'w'a = aw'a'$ $(w \in W)$. Suppose $a' \neq 0$. Then $a'W' \neq 0$ (Lemma B (4)), and so there exists $w_0 \in W$ such that $a'w_0'$ is a non-zero central element. Since $a'w_0'a = aw_0'a'$, we see that $a'w_0'a[a, w] = a[aw_0'a', w] = a[a', w]aw_0' \in C$, and so $a[a, w] \in C(w \in W)$. Furthermore, by making use of $a[a, w'] = 0$, we see that $2a'w_0'a[a, w] = aw_0'a'[a, w] + a[aw_0'a', w] = aw_0'(a[a, w])' = 0$. Hence $a[a, W] = 0$, which is impossible (Lemma B (4)). Next, suppose $a' = 0$. Then $aU'' = (aU')' = 0$ and $aU'S''' = a[U', S'']' \subseteq C$. Since $aU' \neq 0$ (Lemma B (4)), we have $S''' \subseteq C$, and so $aU'S'' = a[U, S'']' \subseteq C$. But this forces a contradiction $S'' \subseteq C$.

(7-U) $\Rightarrow$ (1-U). Suppose, to the contrary, that $S' \nsubseteq C$ (Lemma B (1)). Linearizing the relation $[u, u'] \in C$, we get $[u, v'] - [u', v] \in C$ $(u, v \in U)$. Then, by making use of Jacobi identity, we see that $[u, [u, v']] = [u, [u, v]'] - [u', [u, v]] \in C$. Hence $[w', [w', u]] = [u, [u, w']] - [u + w', [u + w', w']] \in C$, namely $[w', [w', U]] \subseteq C$ for any $w \in W$. But this is impossible (Claim).

(8-U) $\Rightarrow$ (3-S). Since $[S^*, S''] = ([S, S']')^* \subseteq C$, if $S'' \nsubseteq C$ then there holds (5-S) (with respect to δ). Then $S \subseteq C$, a contradiction.

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