Math. J. Okayama Univ. 27 (1985), 19–23

ON GENERALIZATION OF A THEOREM OF POSNER

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Throughout the present paper, R will represent a ring with center C, $d: x \to x'$ a derivation of R, and U a differential ideal of R whose left annihilator $l(U) = 0$. Let $K = \{x \in R \mid x' = 0\}$, and $K_0 = \{x \in R \mid (RxR)' = 0\}$, which is an ideal of R. Let I be the ideal of R generated by R', and $V = U \cap K$.

Our present objective is to prove the following theorems.

Theorem 1. *Let R be a d-semiprime ring such that $[u', u] \in C$ for all $u \in U$. If K_0 is commutative, then $[U, U] \subseteq C$.*

Theorem 2. *Let R be a d-semiprime ring such that $[u', u] \in C$ for all $u \in U$. If $[V, V] \subseteq I$ and $[U, R] \subseteq K$ then R is commutative.*

If R is a semiprime ring then $K_0 \subseteq K$. Furthermore, if R is a prime ring and $d \neq 0$ then K_0 has to be zero (see, e.g., [2, Lemma 1 (3)]), and Theorem 1 deduces a generalization of Posner's theorem [4, Theorem 2] (see Corollary 1).

In advance of proving our theorem, we recall several definitions and preliminary results (see [3, § 3]). We say that R is d-prime provided if J_1, J_2 are differential ideals of R and $J_1J_2 = 0$ then $J_1 = 0$ or $J_2 = 0$, or equivalently, if $x, y \in R$ and $xRy^{(k)} = 0$ for all $k \geq 0$ then $x = 0$ or $y = 0$. If R is d-prime then it is easy to see that R is either of prime characteristic or torsion free. A differential ideal P of R is said to be d-prime if the factor ring R/P is d-prime. We say that R is d-semiprime if the intersection of all d-prime ideals of R is zero, or equivalently, if R is differentially isomorphic to a subdirect sum of d-prime rings. If R is d-semiprime, then $l(U) = 0$ shows that the intersection of all d-prime ideals not including U is zero. If R is d-prime, "$l(U) = 0$" becomes "$U \neq 0$".

Lemma 1. *Let A be a ring with center Z, and S an ideal of A with $l(S) = 0$. If $[S, S] \subseteq Z$ then $[s, x][s, y] = 0$ for any $s \in S$ and $x, y \in A$.*

Proof. Let $s, t, u \in S$, and $x, y \in A$. Since $[s, t][s, u] = [s, ts]u$

$-[s,t]us = u[s,ts]-u[s,t]s = 0$, we see that $[s,t][s,x]u = [s,t][s,xu] -[s,t]x[s,u] = 0$, which implies $[s,t][s,x] = 0$. Hence, $[s,x][s,y]t = [s,x][s,yt]-[s,x]y[s,t] = 0$, which concludes that $[s,x][s,y] = 0$.

Lemma 2. *Let R be a d-prime ring. Suppose that $[u',u] \in C$ for all $u \in U$. Then either $[U,U] \subseteq C$ or $[u',u] = 0$ for all $u \in U$. In case $[U,U] \subseteq C$, $[u,v]^2 = 0$ for all $u, v \in U$.*

Proof. We claim first that $C \subseteq K$ or $[U,U] \subseteq C$. Linearlizing the relation $[u',u] \in C$ $(u \in U)$, we get $[u,v']-[u',v] \in C$ $(u,v \in U)$. Hence, for any $c \in C$, $[u,v]c' = ([u,(vc)']-[u',vc])-([u,v']-[u',v])c \in C$, so that $[[u,v],x]c^{(k)} = 0$ $(x \in R,\ u,v \in U$ and $k \geq 1)$. This implies that either $C \subseteq K$ or $[U,U] \subseteq C$.

If $[U,U] \subseteq C$ then $[u,v]^2 = 0$ by Lemma 1. We assume henceforth that $C \subseteq K$. From the proof of [2, Theorem 1 (2)], we can easily see that $u[u',u]^2 = 0$ $(u \in U)$. Combining this with $[u',u] \in K$, we get $u[u',u]R[u',u]^{(k)} = 0$ $(k \geq 0)$, and therefore $u[u',u] = 0$. Furthermore, this implies that $uR[u',u]^{(k)} = 0$ $(k \geq 0)$. Hence $[u',u] = 0$.

Lemma 3. *Let R be a d-prime ring. If $[U,U] \subseteq C$ and $[U,R] \subseteq K$, then R is commutative.*

Proof. Let $u \in U$, and $x, y \in R$. Then, by Lemma 1, $[u,x]y[u,x] = [u,x][u,yx]-[u,x][u,y]x = 0$, i.e., $[u,x]R[u,x] = 0$, and therefore $[u,x]R[u,x]^{(k)} = 0$ for all $k \geq 0$. Hence $[u,x] = 0$, which proves that $U \subseteq C$. Now, we see that $[x,y]u = [x,yu] = 0$, and therefore $[x,y] = 0$.

Lemma 4. *Let R be a d-semiprime ring such that $[u',u] = 0$ for all $u \in U$.*

(1) *$R' \subseteq C$ and $[R,R] \subseteq K$.*

(2) *$I[R,K] = [R,K]I = 0$ and $R[R,K] \cup [R,K]R \subseteq K$.*

(3) *$I \cap K$ contains no non-zero nilpotent elements.*

Proof. (1) Let P be an arbitrary d-prime ideal of R not including U. Then, in view of [3, Lemma 7], either $R' \subseteq P$ or $[R,R] \subseteq P$, so that $[R',R]' \subseteq P$. Hence $[R',R] = 0$ and $[R,R] \subseteq K$.

(2) Let $x, y \in R$, and $a \in K$. Then, by (1), $x'[y,a] = [y,x'a] = [y,(xa)']-[y,xa'] = 0$ and $(x[y,a])' = x'[y,a] = 0$. This proves that $I[R,K] = 0$ and $R[R,K] \subseteq K$, and similarly $[R,K]I = 0$ and

$[R, K]R \subseteq K$.

(3) Let a be an element of $I \cap K$ such that $a^2 = 0$. Then, for any $x \in R$, (2) shows that $axa = a[x, a] = 0$, which proves that a generates a nilpotent differential ideal of R. Hence $a = 0$.

Lemma 5. *Let R be a d-semiprime ring such that $[u', u] = 0$ for all $u \in U$. If $[V, V] \subseteq I$ then R is commutative.*

Proof. Let $v, w \in V$, and $x, y, z \in R$. By Lemma 4 (2), we have $[v, w]^2 = 0$. Hence, $[v, w] = 0$ by Lemma 4 (3) ; V is commutative. Since both $[x, v]$ and $[x, v]y$ are in V by Lemma 4 (1) and (2), we get $[x, v][y, v] = v[x, v]y - [x, v]vy = 0$. Furthermore, $[x, v]y \cdot [z, v] = [z, v][x, v]y = 0$. This proves that $[x, v]$ generates a nilpotent differential ideal of R. Hence, $[x, v] = 0$; $V \subseteq C$. Now, let $u \in U$. Noting that $[R, U] \subseteq V \subseteq C$ (Lemma 4 (1)), we have $[x, u]^2 = xu[x, u] - u[x, xu] = x[x, u]u - [x, xu]u = 0$. This proves that $[x, u]$ generates a nilpotent differential ideal of R, so that $[x, u] = 0$; $U \subseteq C$. We see therefore that $[x, y]u = [x, yu] = 0$, which implies $[x, y] = 0$.

We are now ready to complete the proof of our theorems.

Proof of Theorem 1. There holds $\bigcap_{\lambda \in \Lambda} P_\lambda = 0$ with d-prime ideals $P_\lambda \not\supseteq U$. We put $\Lambda_1 = \{\lambda \in \Lambda \mid P_\lambda \supseteq R'\}$ and $\Lambda_2 = \{\lambda \in \Lambda \mid P_\lambda \not\supseteq R'\}$. Let D_0 be the ideal of R generated by $[[U, U], R]$. Then, Lemma 2 together with [3, Lemma 7] shows that $D_0 \subseteq P_\lambda$ for all $\lambda \in \Lambda_2$. Hence $D_0' \subseteq P_\lambda$ for all $\lambda \in \Lambda$, and therefore $D_0' = 0$ and $D_0 \subseteq K_0$. By hypothesis, D_0 is then a commutative ideal. Now, let $\mu \in \Lambda_1$. Then $\bar{R} = R/P_\mu$ is a prime ring. If $D_0 \not\subseteq P_\mu$ then $\bar{D}_0$ is a non-zero commutative ideal of the prime ring $\bar{R}$. Hence $\bar{R}$ is commutative by [2, Lemma 1 (1)], which contradicts $\bar{D}_0 \neq 0$. We have thus seen that $D_0 \subseteq P_\lambda$ for all $\lambda \in \Lambda$, namely $D_0 = 0$, which concludes $[U, U] \subseteq C$.

Corollary 1. *Let R be a semiprime ring such that $[u', u] \in C$ for all $u \in U$. If $[K_0, K_0] \subseteq I$ then R is commutative. In particular, if K_0 is commutative then R is commutative.*

Proof. Since R is a d-semiprime ring, the intersection of all d-prime ideals not including U is zero. Hence, by Lemma 2, $[u', u]^2 = 0$, and therefore $[u', u] = 0$ for all $u \in U$. Then, Lemma 4 (2) shows that

$[K_0, K_0][R, K_0] = 0$, and we can easily see that $[K_0, K_0]$ generates a nilpotent ideal of R. Thus K_0 has to be commutative. Now, Theorem 1 shows that $[U, U] \subseteq C$. Then, by Lemma 1, $[u, v]^2 = 0$, and so $[u, v] = 0$ for all $u, v \in U$. Since $[x, v]u + [xu, v] = 0$ for all $x \in R$, we get $U \subseteq C$. Furthermore, for any $x, y \in R$ we have $[x, y]u = [xu, y] = 0$, which implies $[x, y] = 0$.

Given an element x of R and a positive integer k, we denote by $T_k(x)$ the ideal of R generated by $\{x, x', \ldots, x^{(k-1)}\}$. Now, let S be a subset of R. Following [1], we say that d satisfies the condition (F) on S if for each $s \in S$ there exists a positive integer $k = k(s)$ such that $s^{(k)} \in T_k(s)$.

Corollary 2. *Let R be a d-semiprime ring such that $[u', u] \in C$ for all $u \in U$. Suppose that K_0 is commutative. If d satisfies the condition* (F) *on $[U, U]$, then R is commutative. In particular, if for each $u \in U$ there exists a positive integer $k = k(u)$ such that $u^{(k)} \in C$, then R is commutative.*

Proof. It suffices to show that $[U, U] = 0$ (see the proof of Corollary 1). Suppose, to the contrary, that $s = [u_0, v_0] \neq 0$ for some $u_0, v_0 \in U$. In view of Theorem 1, $[U, U] \subseteq C$ and $[u, v]^2 = 0$ for all $u, v \in U$ (Lemma 1). By hypothesis, there exists a positive integer k such that $s^{(k)} \in T_k(s)$. Then $T_k(s)$ is a non-zero nilpotent differential ideal of R, which is a contradiction.

Proof of Theorem 2. Let P be an arbitrary d-prime ideal of R not including U, and $\bar{R} = R/P$. Then, by Lemma 2, either $[\bar{U}, \bar{U}]$ is included in the center of $\bar{R}$ or $[\bar{u}', \bar{u}] = 0$ for all $u \in U$. In case $[\bar{U}, \bar{U}]$ is included in the center of $\bar{R}$, Lemma 3 shows that $\bar{R}$ is commutative ; in particular, $[\bar{u}', \bar{u}] = 0$ for all $u \in U$. This proves that $[u', u] = 0$ for all $u \in U$. Hence, R is commutative by Lemma 5.

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(*Received October 9, 1984*)