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# DIFFERENTIAL RINGS WITH CENTRAL DERIVATIVES OF HIGHER ORDER

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Let $R$ be a differential ring with the derivation $d: x \to x'$, and $Z$ the center of $R$. We say that $R$ is $d$-prime provided if $I$, $J$ are differential ideals of $R$ and $IJ=0$ then $I=0$ or $J=0$, or equivalently, if $x, y \in R$ and $x^{(h)}Ry = 0$ for all $h \geq 0$ then $x=0$ or $y=0$. If $R$ is $d$-prime then it is easy to see that $R$ is either of prime characteristic $p$ or torsion free ($p=\infty$). A differential ideal $P$ of $R$ is said to be $d$-prime if the factor ring $R/P$ is $d$-prime. We say that $R$ is $d$-semiprime if the intersection of all $d$-prime ideals of $R$ is zero, or equivalently, if $R$ is differentially isomorphic to a subdirect sum of $d$-prime rings (see, e.g., [3, § 3]).

The main objective of this paper is to prove the following

**Theorem 1.** *Let $R$ be a $d$-prime ring of characteristic $p$, $S$ a differential subring of $R$, and $n$ a non-negative integer such that $p > n+1$. If $S^{(n)} \neq 0$ and $S^{(n)} \subseteq Z$, then $S \subseteq Z$.*

In advance of proving Theorem 1, we state two lemmas. We can easily see the first one, whose proof may be omitted.

**Lemma 1.** *If a $d$-prime ring $R$ contains a commutative, non-zero differential right (or left) ideal, then $R$ is commutative.*

**Lemma 2.** *Let $R$ be a $d$-prime ring of characteristic $p$, $S$ a differential subring of $R$, and $n$ a positive integer. Suppose $S^{(n)} \subseteq Z$ and $S^{(n-1)} \not\subseteq Z$. If $p > n$ then $S^{(n+1)} = 0$, and if $p > n+1$ then $S^{(n)} = 0$.*

*Proof.* Throughout the proof, $x, y$ will denote arbitrary elements in $S$, and $r$ an arbitrary element in $R$.

Since $0 = [(x^{(n-1)}y^{(n)})^{(n)}, r] = y^{(2n)}[x^{(n-1)}, r]$, we have $(y^{(2n)})^{(h)}R[x^{(n-1)}, r] = 0$ for all $h \geq 0$. Noting that $R$ is $d$-prime and $S^{(n-1)} \not\subseteq Z$, we get $S^{(2n)} = 0$. Suppose now that $S^{(n+k)} = 0$ for some $k > 1$. Then

$$0 = [(x^{(k-2)}y^{(n)})^{(n)}, r] = \sum_{i=0}^{n} \binom{n}{i} y^{(n+i)}[x^{(n+k-i-2)}, r].$$

Since $y^{(n+i)} = 0$ for $i > k-1$ and $x^{(n+k-i-2)} \in Z$ for $i < k-1$, this gives

$\binom{n}{k-1}y^{(n+k-1)}[x^{(n-1)}, r] = 0$, and so $y^{(n+k-1)}R[x^{(n-1)}, r] = 0$. Hence $S^{(n+k-1)} = 0$, which proves that $S^{(n+1)} = 0$. Then, we can easily see that no non-zero element in $S^{(n)}$ is a zero-divisor in $R$. Since

$$0 = [(x^{(n-1)}x)^{(n)}, r] = (n+1)x^{(n)}[x^{(n-1)}, r],$$

if $p > n+1$ then we get $S^{(n)} = 0$, by Brauer's trick.

*Proof of Theorem* 1. In case $n = 0$, there is nothing to prove. If $n > 0$, Lemma 2 shows that $S^{(n-1)} \subseteq Z$. Hence $S \subseteq Z$, by induction.

As an application of Theorem 1, we obtain the following

**Theorem 2.** *Let $R$ be a d-prime ring of characteristic $p$, $U$ a non-zero differential ideal of $R$, and $n$ a non-negative integer such that $p > n+1$. If $R^{(n)} \neq 0$ and $U^{(n)} \subseteq Z$, then $R$ is commutative.*

*Proof.* We can apply the argument employed in the proof of [2, Theorem] to see that $R^{(n)} \neq 0$ implies $U^{(n)} \neq 0$. Then $U \subseteq Z$ by Theorem 1, and hence $R$ is commutative by Lemma 1.

**Corollary 1** (cf. [3, Corollary 1]). *Let $R$ be a d-prime ring of characteristic $p$, $U$ a non-zero differential ideal of $R$, and $n$ an even positive integer such that $p > n+1$. If $R^{(n-1)} \neq 0$ and $U^{(n)} \subseteq Z$, then $R$ is commutative.*

*Proof.* By making use of the same method as in the proof of [1, Theorem], we can prove that $R^{(n)} \neq 0$. Hence, $R$ is commutative by Theorem 2.

**Theorem 3.** *Let $n$ be a positive integer. Let $R$ be a $(n+1)!$-torsion free d-semiprime ring, and $U$ a differential ideal of $R$ with $l(U) = 0$. If $K_n = \{x \in R \mid (RxR)^{(n)} = 0\}$ is commutative and $U^{(n)} \subseteq Z$ then $R$ is commutative.*

*Proof.* As is easily seen, $\bigcap_{\lambda \in \Lambda} P_\lambda = 0$ with $d$-prime ideals $P_\lambda$ such that $U \not\subseteq P_\lambda$ and $R/P_\lambda$ is of characteristic $> n+1$. Put $\Lambda_1 = \{\lambda \in \Lambda \mid R^{(n)} \not\subseteq P_\lambda\}$, and let $D$ be the commutator ideal of $R$ (note that $D$ is a differential ideal of $R$). If $\lambda \in \Lambda_1$ then Theorem 2 proves that $D \subseteq P_\lambda$. Since $(RDR)^{(n)} \subseteq P_\lambda$ for all $\lambda \in \Lambda \backslash \Lambda_1$, we see that $(RDR)^{(n)} \subseteq \bigcap_{\lambda \in \Lambda} P_\lambda = 0$. This means $D \subseteq K_n$, and so $D$ is a commutative ideal. Now, by Lemma 1, we can easily see that $D \subseteq \bigcap_{\lambda \in \Lambda} P_\lambda = 0$. This proves that $R$ is commutative.

By making use of Corollary 1 instead of Theorem 2, the proof of Theorem 3 gives the following

**Corollary** 2 (cf. [3, Theorem 3]). *Let $n$ be an even positive integer. Let $R$ be a $(n+1)!$-torsion free d-semiprime ring, and $U$ a differential ideal of $R$ with $l(U)=0$. If $K_{n-1}$ is commutative and $U^{(n)} \subseteq Z$, then $R$ is commutative.*

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