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ON CERTAIN PERIODIC RINGS

Dedicated to Professor Katsumi Numakura on his 60th birthday

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Let R be a ring (not necessarily with 1), N the set of nilpotent elements in R, and N^* the subset of N consisting of all a with $a^2 = 0$. Let D be the set of right and left zero-divisors in R. Given a positive integer $n > 1$, we set $E_n = \{x \in R \mid x^n = x\}$; in particular, $E = E_2$. As is well known, if R is periodic then every element x in R can be written in the form $x = e + a$, where $a \in N$ and $e \in E_n$ for some n.

In this paper, we prove the following theorem, which includes all the results in [1].

Theorem. *Let R be a periodic ring with N^* commutative.*

(1) *Then N coincides with the Jacobson radical of R and $\overline{R} = R/N$ is a subdirect sum of fields.*

(2) *Let $n > 1$ be a fixed positive integer. If every element d in D can be written in the form $d = e + a$, where $a \in N$ and $e \in E_n$, then $\overline{R}$ is either a field or $\overline{E}_n$.*

Proof. (1) Let J be the Jacobson radical of the periodic ring R, which is obviously a nil ideal. First, we claim that every idempotent $\bar{e}$ of $\overline{R} = R/J$ is central. Since J is a nil ideal, we may assume from the beginning that e is an idempotent of R. By hypothesis, $eR(1-e)\cdot(1-e)Re = (1-e)Re\cdot eR(1-e)$, and so $eR(1-e)Re = 0$ (where 1 is used formally). Hence, by the semiprimeness of $\overline{R}$, we get $\bar{e}\overline{R}(1-\bar{e}) = 0$, and therefore $\bar{e}\bar{x} = \bar{e}\bar{x}\bar{e}$ for all $x \in R$. Furthermore, $\bar{e}\overline{R}(1-\bar{e})\overline{R} = 0$ yields $(1-\bar{e})\overline{R}\bar{e} = 0$, and so $\bar{x}\bar{e} = \bar{e}\bar{x}\bar{e}$ for all $x \in R$. Thus we have seen that $\bar{e}$ is central. Now, it is easy to see that every nilpotent element of $\overline{R}$ generates a nil right ideal. Hence N coincides with J. As is easily seen, for any element $x \in R$ there exists a non-negative integer k such that $x - x^{k+2} \in N$. Hence $\overline{R}$ is a subdirect sum of fields, by Jacobson's commutativity theorem.

(2) Let x be an arbitrary element of R. If $x \notin D$ then $x = x^{k+2}$ with some non-negative integer k. Then $e = x^{k+1}$ is a right or left unity of R, and therefore $\bar{e}$ is the unity of the commutative ring $\overline{R}$ and $\bar{x}$ is a unit (see (1)). On the other hand, if $x \in D$ then $\bar{x}^n = \bar{x}$, by hypothesis.

In case $R = D$, it is clear that $\overline{R} = \overline{E}_n$. In what follows, we consider the case that $R \neq D$. Then, by the above claim, $\overline{R}$ has the unity $\bar{e}$. It suffices therefore to show that if $\overline{R}$ contains a non-unit $\bar{x} \neq 0$ then $\overline{R} = \overline{E}_n$. Actually, by the above claim, $\bar{f} = \bar{x}^{n-1}$ is an idempotent with $\bar{x}\bar{f} = \bar{x}$ and $\overline{R} = \bar{f}\overline{R} \oplus (\bar{e}-\bar{f})\overline{R}$, so that $\overline{R} = \overline{E}_n$.

Combining the proof of Theorem (2) with [3, Proposition 2], we can easily see the following.

Corollary. *Let R be a periodic ring with N^* commutative. If D is included in the subring $\langle E \cup N \rangle$ generated by $E \cup N$, then $\overline{R}$ is either a field or a subdirect sum of finite prime fields.*

Remark. Following [2], a ring R is called an I-ring (resp. I'-ring) if every element of R can be written as a product of elements in E (resp. $E \cup N$). Now, let R be a (not necessarily periodic) I'-ring with N^* commutative. Then, the argument employed in the proof of Theorem (1) enables us to see that every idempotent in the factor ring of R modulo its prime radical P is central. If $E \neq 0$, then R/P is an I-ring by [2, Lemma 1]. Hence R/P is a Boolean ring, and N coincides with P. Also, if N is multiplicatively closed (especially, if N is commutative) then N forms an ideal of R.

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