Math. J. Okayama Univ. 27 (1985), 1 — 3

ON COMMUTATIVITY AND STRUCTURE OF PERIODIC RINGS

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A ring R is called periodic if for each $x \in R$ there exist distinct positive integers m, n for which $x^n = x^m$. Years ago Herstein [8] proved that periodic rings with all nilpotent elements central must be commutative ; more recently, I showed in [3] that for periodic R with commuting nilpotent elements, the commutator ideal is nil and the nilpotent elements form an ideal. Since then several authors [1, 2, 6, 9, 11] have found sufficient conditions for commutativity of periodic rings with commuting nilpotent elements. The purpose of this note is to provide two more conditions of this type, the second being motivated by Theorem 1 of [11].

Before stating the results, we fix some notation and quote some background results. We shall denote by N the set of nilpotent elements of R, by Z the center of R, and by P the set of potent elements of R – that is, the set of $x \in R$ for which there exists an integer $n = n(x) > 1$ such that $x^n = x$. If $P = R$, we shall call R a J-ring; and we shall on occasion invoke Jacobson's "$x^n = x$ theorem", which asserts that J-rings are commutative. We shall also use the following results on periodic rings R, proofs of which may be found in [4]:

(i) for each $x \in R$, some power of x is idempotent ;

(ii) for each $x \in R$, there exists an integer $n = n(x) > 1$ for which $x - x^n \in N$;

(iii) if I is an ideal of R and $x + I$ is a nilpotent element of R/I, there exists u ∈ N such that $x \equiv u \pmod{I}$.

Our commutativity results are expressed by the following two theorems.

Theorem 1. *Let R be a periodic ring in which N is commutative. Then R is commutative if there exists a prime p such that for each $x \in R$, there is an integer $n = n(x) \geq 1$ for which $x^{p^n} \in Z$.*

Theorem 2. *Let R be a periodic ring with N commutative ; and suppose that (†) each element x has a unique representation in the form $a + u$, where a*

*Supported by the Natural Sciences and Engineering Research Council of Canada, Grant No. A3961.

$\in P$ and $u \in N$. Then R is commutative.

Proof of Theorem 1. In view of (iii), we may assume that R is subdirectly irreducible, in which case R contains no non-zero central idempotents except possibly 1. We assume also that $R \neq N$. Noting (i) and observing that all idempotents are central, we conclude that R has 1 and each non-nilpotent element has a power equal to 1. The periodicity of R now guarantees the existence of a prime q and a positive integer α for which $q^{\alpha}R = \{0\}$; and since Theorem 4 of [5] yields commutativity of R if $q \neq p$, we may assume henceforth that $q = p$.

As we observed earlier, the hypothesis that N is commutative forces N to be an ideal ; and (ii) implies that $\bar{R} = R/N$ is a J-ring. Thus, if $a \notin N$, the subring of $\bar{R}$ generated by $a+N$ is a finite field GF (p^t). Taking s such that $a^{p^s} \in Z$ and noting that $a^{p^{st}} - a \in N$, we see that $[a, u] = 0$ for all $u \in N$. Thus $N \subseteq Z$, and R is commutative by Herstein's theorem.

Proceeding now to the proof of Theorem 2, we note that the commutativity of R follows at once from the commutativity of J-rings and the following, perhaps surprising, structural result.

Theorem 3. *Let R be any periodic ring satisfying* (†). *Then N and P are both ideals, and $R \cong P \oplus N$.*

Proof. Let e be an idempotent, and x an arbitrary element of R. The element $e+ex-exe$, which is easily verified to be idempotent, has two obvious representations in form $a+u$, with $a \in P$ and $u \in N$; one is obtained by taking $a = e+ex-exe$ and $u = 0$, the other by taking $a = e$ and $u = ex-exe$. Applying (†) shows that $ex-exe = 0$; and since a similar argument gives $xe-exe = 0$, we see that idempotents in R must be central.

Suppose now that S is a subring of R having a multiplicative identity, and that $w \in N \cap S$. Since $1+w$ is invertible, it is clearly in P; hence it has two representations of form $a+u$ with $a \in P$ and $u \in N$, one being $a = 1+w$ and $u = 0$, the other $a = 1$ and $u = w$. Thus, (†) implies that S is reduced (i.e. has no non-zero nilpotent elements). Taking S to be a subring of form eR, where e is any non-zero idempotent of R, we see that eR is reduced ; and it is not difficult to argue that there is a reduced ideal containing all idempotents of R. By [10, Theorem 1] it follows that $R \cong P \oplus N$.

Remark. In the statement of Theorem 1, the prime p cannot be replaced by an arbitrary positive integer k. For example, consider a non-commutative finite ring with 1 in which $N^2 = 0$ and every non-nilpotent element is invertible. (The rings discussed by Corbas in [7] provide examples.) Clearly there exists an integer $k \geq 2$ such that for each $x \in R$, either $x^k = 1$ or $x^k = 0$; hence $x^k \in Z$ for all $x \in R$.

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(*Received August 4, 1984*)