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# RIEMANNIAN MANIFOLDS ADMITTING COMMUTATIVE KILLING VECTOR FIELDS

Dedicated to Professor Shigeo Sasaki on his 70th birthday

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## 1. Introduction

Let $M$ be a compact $C^\infty$-Riemannian manifold. Suppose that $M$ admits commutative and linearly independent Killing vector fields $X$ and $Y$. In this paper, we shall study the following problems ; What kind of restrictions to Riemannian structure of $M$ should be imposed by the existence of such vector fields ? Conversely, does the Riemannian structure of $M$ give any restriction to such vector fields $X$ and $Y$ ?

Let $M$ be a compact Riemannian manifold of non-positive sectional curvature. The following fact is known by Lawson-Yau [5] ; There exists an abelian subgroup of rank $k$ in the fundamental group $\pi_1(M)$ if and only if there exists a flat $k$-torus isometrically and totally geodesically immersed in $M$. In particular, this shows Preissman's result [6] in the case when $M$ has negative sectional curvature, that is, every abelian subgroup of $\pi_1(M)$ is rank one (i.e., cyclic). Now, let $\widetilde{M}$ be a covering space of $M$. A subgroup of $\pi_1(M)$ determined by the covering map is identified with a group $\Gamma$ of isometries of $\widetilde{M}$, so-called the group of all deck transformations. From this point of view, we shall consider our problems in the case that $\widetilde{M}$ has no assumption on sectional curvature and that $\Gamma$ is replaced by a torus (or $\mathbf{R}^2$) group acting on $\widetilde{M}$. Therefore, our problem may be seen to be a trial at the problem of Lawson-Yau type concerning an abelian subgroup of the fundamental group. In fact, our Corollary 2 is an extention of the fact "Every abelian subgroup in the fundamental group of the spherical space form is cyclic".

Before we state our results, we give some typical examples of Riemannian manifolds with commutative and linearly independent Killing vector fields.

**Example 1.** Let $S^{2m}$ $(m \geqq 2)$ be the standard $2m$-sphere. $S^{2m}$ has commutative and linearly independent Killing vector fields. In this case, every Killing vector field on $S^{2m}$ has a zero. Generally, if $M$ is an even dimensional compact Riemannian manifold with positive sectional curvature, every Killing vector field on $M$ has a zero [1 or 4, Cor. 5.7].

**Example 2.** Let $S^3 = \{(z_1, z_2) : z_i \in \mathbf{C}$ and $|z_1|^2 + |z_2|^2 = 1\}$ be the standard 3-sphere. Let $\{\phi_t\}_{t\in\mathbf{R}}$ and $\{\psi_s\}_{s\in\mathbf{R}}$ be 1-parameter groups of the unitary group $U(2)$ defined by

$$\phi_t = \begin{bmatrix} \exp(\sqrt{-1}\,p_1 t) & 0 \\ 0 & \exp(\sqrt{-1}\,p_2 t) \end{bmatrix},$$
$$\psi_s = \begin{bmatrix} \exp(\sqrt{-1}\,q_1 s) & 0 \\ 0 & \exp(\sqrt{-1}\,q_2 s) \end{bmatrix},$$

where $p_i$ and $q_i$ are non-zero real numbers such that $(p_1, p_2)$ and $(q_1, q_2)$ are linearly independent. For each $t \in \mathbf{R}$, $\phi_t$ acts isometrically on $S^3$ by $\phi_t[(z_1, z_2)] = (\exp(\sqrt{-1}\,p_1 t)\cdot z_1, \exp(\sqrt{-1}\,p_2 t)\cdot z_2)$. Then, Killing vector fields $X$ and $Y$ generated by $\phi_t$ and $\psi_s$ are commutative and linearly independent. In this case, we have $\{\phi_t[(1,0)] : t \in \mathbf{R}\} = \{\psi_s[(1,0)] : s \in \mathbf{R}\}$.

**Example 3.** Let $G$ be a compact Lie group of rank $m$ $(m \geqq 2)$ and $g$ a left and right invariant Riemannian metric on $G$. We take vectors $\mathbf{x}$ and $\mathbf{y}$ of its Lie algebra $\mathfrak{g}$ such that $[\mathbf{x}, \mathbf{y}] = 0$, $\|\mathbf{x}\| = \|\mathbf{y}\| = 1$ and $g(\mathbf{x}, \mathbf{y}) = 0$. For the vector $\mathbf{x}$, we define a 1-parameter group $\phi_t$ of isometries by $\phi_t(g) = \exp(t\mathbf{x})\cdot g$ for $g \in G$. We denote by $X$ the Killing vector field generated by $\phi_t$. Similarly, we have a 1-parameter group $\psi_s$ of isometries and a Killing vector field $Y$ from $\mathbf{y}$. $X$ and $Y$ are commutative and linearly independent. Then, we can define an isometric immersion $x$ of $\mathbf{R}^2$ into $M$ by $x(t, s) = \exp(t\mathbf{x} + s\mathbf{y})$, where $\mathbf{R}^2$ is the real 2-plane with canonical metric. This immersion is an orbit under $\phi_t$ and $\psi_s$ in $G$ through the unit, and we have

$$K(X \wedge Y)_{x(t,s)} = 0,$$

where $K(X \wedge Y)_{x(t,s)}$ is a sectional curvature of the plane spanned by $X_{x(t,s)}$ and $Y_{x(t,s)}$ [see § 2, Ex.4].

Our result is that we have at least one of the characteristics stated in above examples with respect to either manifold $M$ or vector fields $X$ and $Y$. We denote by $g$ the Riemannian metric on $M$, by $\|u\|$ the norm of vector $u$ and by $T_p(M)$ the tangent space to $M$ at a point $p$. For vectors $u$ and $v$ of $T_p(M)$, we denote by $K(u \wedge v)$ the sectional curvature of the plane in $T_p(M)$ spanned by $u$ and $v$. Let $\nabla$ be the covariant differentiation with respect to $g$. The curvature tensor $R(u, v)w$ on $M$ is given by $R(u, v)w = \nabla_u \nabla_v w - \nabla_v \nabla_u w - \nabla_{[u,v]}w$.

**Theorem 1.** *Let* $(M, g)$ *be a compact* $C^\infty$*-Riemannian manifold, and* $X$ *and* $Y$ *Killing vector fields on* $M$ *with* $[X, Y] = 0$. *Let* $\{\tilde{\phi}_t\}_{t\in\mathbf{R}}$ (*resp.* $\{\tilde{\psi}_s\}_{s\in\mathbf{R}}$) *be a* 1*-parameter group of isometries generated by* $X$ (*resp.* $Y$). *Then, we have at least one of the following results* (1), (2) *and* (3).

(1) *There exists a point* $p$ *of* $M$ *such that either* $X_p = 0$ *or* $Y_p = 0$ *holds.*

(2) *There exists a common integral curve of* $X$ *and* $Y$ (*as an image of the curves*).

(3) *There exist an isometric immersion* $x$ *of* $\mathbf{R}^2$ *into* $M$ *and vector fields* $Z$ *and* $V$ *defined on* $x(\mathbf{R}^2)$ *such that* $Z \in x_*(T(\mathbf{R}^2))$, $\nabla_Z Z = 0$, *and* $K(Z \wedge V) = 0$ *and* $Z \wedge V \neq 0$ *on* $x(\mathbf{R}^2)$. *Furthermore, the immersion* $x$ *and vector fields* $Z$ *and* $V$ *satisfy the following conditions* (a) *and* (b).

(a) $K[x_*(\partial/\partial t) \wedge x_*(\partial/\partial s)]$ *is non-negative constant on* $x(\mathbf{R}^2)$, *where* $(t, s)$ *is the canonical coordinate system in* $\mathbf{R}^2$.

(b) *The image* $x(\mathbf{R}^2)$ *and vector fields* $Z$ *and* $V$ *are preserved by the action of group* $\{\tilde{\phi}_t, \tilde{\psi}_r : t, r \in \mathbf{R}\}$.

**Remark 1.1.** In (a) of the statement (3) in Theorem 1, if $K[x_*(\partial/\partial t) \wedge x_*(\partial/\partial s)] = 0$, then we have $V \in x_*(T(\mathbf{R}^2))$.

By Theorem 1, if $X$ and $Y$ are linearly independent at each point of $M$, we have only the case (3) in Theorem 1.

**Theorem 2.** *In Theorem* 1, *we assume further that* $M$ *has non-negative sectional curvature. Then, we have at least one of the following results* (1), (2) *and* (3).

(1) *There exists a point* $p$ *of* $M$ *such that either* $X_p = 0$ *or* $Y_p = 0$ *holds.*

(2) *There exists a common integral curve of* $X$ *and* $Y$.

(3) *There exist an isometric immersion* $x$ *of* $\mathbf{R}^2$ *into* $M$ *and vector fields* $Z$ *and* $V$ *defined on* $x(\mathbf{R}^2)$ *such that* $Z \in x_*(T(\mathbf{R}^2))$, $\nabla_Z Z = 0$, $\nabla_Z V = 0$, *and* $R(V, Z)Z = R(Z, V)V = 0$ *and* $Z \wedge V \neq 0$ *on* $x(\mathbf{R}^2)$. *Furthermore, the immersion* $x$ *and vector fields* $Z$ *and* $V$ *satisfy the conditions* (a) *and* (b) *of the statement* (3) *in Theorem* 1.

By the difference between Lawson-Yau's result and (3) of Theorem 2, it seems to be difficult, even if the manifold has non-negative sectional curvature, to find a direct relation between higher dimensional space of commutative Killing vector fields and a geometric structure on $M$ [see § 2, Ex. 6].

**Corollary 1.** *In Theorem 1, we assume further that $M$ has positive sectional curvature. Then, we have at least the following* (1) *or* (2).

(1) *There exists a point $p$ of $M$ such that either $X_p = 0$ or $Y_p = 0$ holds.*

(2) *There exists a common integral curve of $X$ and $Y$.*

Corollary 1 was given Takagi [10] and Sugahara [8] in somewhat weak conclusion, that is, $X$ and $Y$ are linearly dependent at some point of $M$. But, we can draw the following Corollary 2 from our statement of Corollary 1. This corresponds to Preissman's result in positive curvature case.

For a Riemannian manifold $M$, we denote by $\widetilde{M}$ the universal covering space of $M$. Then, we have $M = \Gamma \backslash \widetilde{M}$, where $\Gamma$ is the group of all deck transformations. And the fundamental group $\pi_1(M)$ of $M$ is identified with $\Gamma$. For a subgroup $A$ of $\pi_1(M)$, we denote by $\widetilde{A}$ the subgroup of $\Gamma$ corresponding to $A$.

**Corollary 2.** *Let $M$ be a compact Riemannian manifold with positive sectional curvature. Let $A$ be an abelian subgroup of $\pi_1(M)$. Assume that $\widetilde{A}$ is contained in a subgroup of isometric group generated by several Killing vector fields on $\widetilde{M}$, which are commutative to each other. Then, $A$ is cyclic.*

**Remark 1.2.** (1) Let $M$ be a compact Riemannian manifold with positive sectional curvature. It is known by Synge [9 or 4, Th. 5.6] that, if $M$ is even dimensional and orientable, $M$ is simply connected. Therefore, if $M$ is non-orientable, its double covering space is simply connected. If $M$ is odd dimensional, then $M$ is orientable, i. e., each deck transformation of $\widetilde{M}$ is an orientation preserving map.

(2) By the assumption for $\widetilde{A}$ in Corollary 2, we only consider the case when $M$ is odd dimensional [see Ex. 1].

Let $M$ be a spherical space form of dimension $n$. Then, it is known that any abelian subgroup $A$ of $\pi_1(M)$ is cyclic, by the classification of spherical space forms due to Wolf [11]. In this case, if $n =$ odd, the condition that $A$ is abelian implies simultaneously our assumption for $\widetilde{A}$ in Corollary 2. In fact, we have $\Gamma \subset SO(n+1)$ by Synge. Furthermore, since $A$ is abelian, all elements of $\widetilde{A}$ are simultaneously diagonalizable by an orthogonal matrix $P$. So, we can take some Killing vector fields satisfying the assumption in Corollary 2.

In § 2, we shall give a full detail of our results and give several examples in connection with them. Also we shall prove Corollary 2 there. In § 3, we shall prove the results in § 2.

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## 2. Detail of results and more examples

Theorems stated in introduction can be devided into several propositions. We use same notations as in introduction. We assume that $X$ and $Y$ do not have zero on $M$. Let $\bar{Y}$ be a vector field on $M$ defined by

$$\bar{Y} = Y - [g(Y, X)/\|X\|^2]X.$$

There exists a point $p$ of $M$ such that the square norm $\|\bar{Y}\|^2$ of $\bar{Y}$ is critical at $p$. Then, the value $\|\bar{Y}\|^2(p)$ is either zero or non-zero.

**Proposition 1.** *If $\bar{Y}_p = 0$, then two integral curves of $X$ and $Y$ through $p$ coincide (as the image of curves).*

We assume $X_p \neq 0$, $Y_p \neq 0$ and $\bar{Y}_p \neq 0$. Then, put

$$Z = Y - [g(Y, X)/\|X\|^2](p)X.$$

The vector field $Z$ is also Killing. Let $\{\phi_t\}_{t\in\mathbf{R}}$ (resp. $\{\psi_s\}_{s\in\mathbf{R}}$) be a 1-parameter group of isometries generated by $X/\|X\|(p)$ (resp. $Z/\|Z\|(p)$). We define an immersion $x$ of $\mathbf{R}^2$ into $M$ by

$$x(t, s) = \phi_t[\psi_s(p)].$$

**Proposition 2.** *The immersion $x$ is isometric and*

$$g(R(X, Z)Z, X) = g(\nabla_Z X, \nabla_Z X) \quad \text{on } x(\mathbf{R}^2).$$

By Proposition 2, we see that $K[x_*(\partial/\partial t) \wedge x_*(\partial/\partial s)]$ is non-negative constant on $x(\mathbf{R}^2)$. Let $\eta_0 : s \to x(s)$, $s \in \mathbf{R}$, be an integral curve of

$Z/\|Z\|(p)$ through $p$, i.e., $x(s)=\psi_s(p)$. Then, the curve $\eta_0$ is a geodesic in $M$ (see § 3, Prop. 6 and Lemma 3.1). Let $(\eta_0)_s^0$ denote the parallel translation from $T_p(M)$ onto $T_{x(s)}(M)$ along the curve $\eta_0$.

**Lemma.** *There exist a system* $\{u_1, v_1, \cdots, u_k, v_k, e\}$ *of vectors of* $T_p(M)$ *and non-zero real numbers* $\{p_1, \cdots, p_k\}$ *such that*

(a) $$(X)_{x(s)} = \sum_{i=1}^{k} [\cos p_i s(\eta_0)_s^0 u_i + \sin p_i s(\eta_0)_s^0 v_i] + (\eta_0)_s^0 e \quad \textit{for } s \in \mathbf{R},$$

(b) *The system* $\{Z_p, u_1, v_1, \ldots, u_k, v_k, e\}$ *is orthogonal and* $\|u_i\| = \|v_i\|$ $(i=1,2,\cdots,k)$.

In Lemma, put

$$(X_1)_{x(s)} = \sum_{i=1}^{k} [\cos p_i s(\eta_0)_s^0 u_i + \sin p_i s(\eta_0)_s^0 v_i].$$

If $(X_1)_p = 0$, then we can take $X$ as $V$ stated in Theorem 1. In this case, $V$ is tangent to the image $x(\mathbf{R}^2)$.

**Proposition 3.** *If* $(X_1)_p = 0$, *then* $(X)_{x(s)} = (\eta_0)_s^0 X_p$ *and* $K(Z \wedge X) = 0$ *on* $\eta_0$.

If $(X_1)_p \neq 0$, then $K(Z \wedge X)_{x(t,s)}$ is positive constant on $x(\mathbf{R}^2)$ by Proposition 2.

**Proposition 4.** *Take a point* $p$ *at which* $\|\bar{Y}\|$ *attains a local minimum. Assume* $(X_1)_p \neq 0$ *and* $e \neq 0$ *in Lemma. Then,* $K(Z \wedge (\eta_0)_s^0 e)$ *is non-positive constant along* $\eta_0$.

**Remark 2.1.** In the case of Proposition 4, we can take, as $V$ stated in Theorem 1, some linear combination of $X$ and $(\phi_t)_*[(\eta_0)_s^0 e]$. And, as $V$ stated in Theorem 2, we can take $(\phi_t)_*[(\eta_0)_s^0 e]$.

**Proposition 5.** *Take a point* $p$ *at which* $\|\bar{Y}\|$ *attains a local minimum. Assume* $e=0$ *in Lemma. Then, we have that* $K(Z \wedge W)$ *is negative constant along* $\eta_0$, *where*

$$(W)_{x(s)} = \sum_{i=1}^{k} p_i^{-1} [\sin p_i s(\eta_0)_s^0 u_i - \cos p_i s(\eta_0)_s^0 v_i].$$

**Remark 2.2.** (a) In the case of Proposition 5, we can take, as $V$ stated in Theorem 1, some linear combination of $X$ and $(\phi_t)_*W$.

(b) In Proposition 5, we have $(\nabla_Z W)_{x(s)} = \| Z \|(p) X_{x(s)}$.

We note that, if $M$ has non-negative sectional curvature, $K(Z \wedge V) = 0$ implies that $R(V, Z)Z = R(Z, V)V = 0$.

**Corollary 2.** Let $M$ be a compact Riemannian manifold with positive sectional curvature. Let $A$ be an abelian subgroup of $\pi_1(M)$. Assume that $\widetilde{A}$ is contained in a subgroup of isometric group generated by several Killing vector fields on $\widetilde{M}$, which are commutative to each other. Then, $A$ is cyclic.

*Proof of Corollary* 2. Take non-unit elements $\phi$ and $\psi$ of $\widetilde{A}$. By the assumption, we have $\phi_{t_0} = \phi$ and $\psi_{s_0} = \psi$, where $\{\phi_t\}_{t\in\mathbf{R}}$ and $\{\psi_s\}_{s\in\mathbf{R}}$ are 1-parameter groups generated by some commutative Killing vector fields $X$ and $Y$, respectively. Since $\phi$ and $\psi$ have no fixed point, by Corollary 1 there exists a point $p$ of $\widetilde{M}$ such that

$$\{\phi_t(p) : t \in \mathbf{R}\} = \{\psi_s(p) : s \in \mathbf{R}\}.$$

Furthermore, since $\phi$ and $\psi$ have finite order, the set $\{\phi_t(p) : t \in \mathbf{R}\}$ is circle $S^1$. The group $\langle \phi, \psi \rangle$ generated by $\phi$ and $\psi$ is a discrete subgroup acting isometrically on $S^1$. So, the group $\langle \phi, \psi \rangle$ acting on $S^1$ is cyclic. Since $\phi$ and $\psi$ are deck transformations, the group $\langle \phi, \psi \rangle$ acting on $\widetilde{M}$ is also cyclic. If we take non-unit elements $\phi_1$, $\phi_2$ and $\phi_3$ of $\widetilde{A}$, then we have $\langle \phi_1, \phi_2 \rangle = \langle \phi \rangle$ for some $\phi$ of $\widetilde{A}$, by the above argument. Therefore, we have $\langle \phi_1, \phi_2, \phi_3 \rangle = \langle \phi, \phi_3 \rangle = \langle \psi \rangle$ for some $\psi$ of $\widetilde{A}$. Since $\widetilde{A}$ is finite, we can show that $\widetilde{A}$ is cyclic.

*q.e.d.*

We give several examples of Riemannian manifolds with commutative Killing vector fields in connection with the above propositions. The following example is connected with Proposition 3.

**Example 4.** Let $M = K/H$ be a naturally reductive homogeneous space with an $ad(H)$-invariant decomposition $\mathfrak{k} = \mathfrak{h} + \mathfrak{m}$, and $K$-invariant Riemannian metric $g$. Let $\pi: K \to M = K/H$ be the natural projection and put $\pi(H) = o$. We identify $\mathfrak{m}$ with $T_o(M)$ by $\pi$. Let $B$ be a bilinear form on $\mathfrak{m}$ which corresponds to $g$. Suppose that there exist vectors $\mathbf{x}$ and $\mathbf{y}$ of $\mathfrak{m}$ such that $[\mathbf{x}, \mathbf{y}] = 0$, $\|\mathbf{x}\|(o) = \|\mathbf{y}\|(o) = 1$ and $B(\mathbf{x}, \mathbf{y}) = 0$. Then, we define an immersion $x$ of $\mathbf{R}^2$ into $M$ by $x(t, s) = \pi(\exp t\mathbf{x} \cdot \exp s\mathbf{y})$. This immersion $x$ is an orbit through the point $o$ under 1-parameter groups generated by some Killing vector fields $X$ and $Y$ such that $[X, Y] = 0$. Furthermore, $x$ is isometric and $K(X \wedge Y)_{x(t,s)} = 0$. In particular, $x$ is totally geodesic.

In fact, for $k \in K$ and $\mathbf{z} \in \mathfrak{m}$, let $Z_{\pi(k)}$ be a vector tangent to a curve $\pi(\exp t\mathbf{z} \cdot k)$, $t \in \mathbf{R}$, at $\pi(k)$. Then the vector field $Z$ on $M$ is Killing. Furthermore, the curve $\pi(\exp t\mathbf{z})$, $t \in \mathbf{R}$, is a geodesic [cf. 3, Th.2.9 and Th.3.3]. Take $\mathbf{x}$ and $\mathbf{y}$ as above, then $[X, Y] = 0$ and

$$K(X \wedge Y)|_o = K(\mathbf{x} \wedge \mathbf{y})|_o = g(R(\mathbf{x},\mathbf{y})\mathbf{y},\mathbf{x})|_o$$
$$= (1/4)B([\mathbf{x},\mathbf{y}]_{\mathfrak{m}}, [\mathbf{x},\mathbf{y}]_{\mathfrak{m}}) - B([[\mathbf{x},\mathbf{y}]_{\mathfrak{h}}, \mathbf{y}], \mathbf{x}) = 0 \text{ [cf. 3, Th.3.4]}.$$

From $(\nabla_{X+Y}(X+Y))_o = (\nabla_X X)_o = (\nabla_Y Y)_o = 0$, the immersion $x$ is totally geodesic. Furthermore, we can see that the length of $Y - [g(Y,X)/\|X\|^2]X$ is critical at $o$.

The following example is connected with Proposition 5.

**Example 5.** Let $\mathbf{R}^3 = \{(x_1, x_2, x_3) : x_i \in \mathbf{R}\}$. We define a Riemannian metric $g$ on $\mathbf{R}^3$ by

$$g = \begin{bmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{bmatrix} = \begin{bmatrix} 1+\zeta\cos x_3 & \zeta\sin x_3 & 0 \\ \zeta\sin x_3 & 1-\zeta\cos x_3 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

where $g_{ij} = g(e_i, e_j)$, $e_i = \partial/\partial x_i$ and $0 < \zeta < 1$. Then, $e_1$ and $e_2$ are Killing vector fields on $\mathbf{R}^3$. Put $\bar{Y} = e_2 - (g_{12}/g_{11})e_1$. Then, $\|\bar{Y}\|$ attains the minimum on the plane $\{(x_1, x_2, 0)\}$ and $(\bar{Y})_{(x_1,x_2,0)} = (e_2)_{(x_1,x_2,0)}$. Therefore, an isometric immersion $x$ of $\mathbf{R}^2$ into $\mathbf{R}^3$ is given by

$$x(t, s) = (t/\sqrt{1+\zeta},\ s/\sqrt{1-\zeta},\ 0).$$

As for the sectional curvature of the plane spanned by $e_1$ and $e_2$, we have $K(e_1 \wedge e_2)_{(x_1,x_2,x_3)} = \zeta^2/[4(1-\zeta^2)]$. And we have (see Remark 2.2,(b))

$$(\nabla_{e_2} e_3)_{(x_1,x_2,0)} = \zeta/[2(1+\zeta)](e_1)_{(x_1,x_2,0)},$$
$$K(e_2 \wedge e_3)_{(x_1,x_2,0)} = -\zeta(\zeta+2)/[4(1-\zeta^2)].$$

As for Lemma, we have

$$(\nabla_{e_2} e_1)_{(x_1,x_2,0)} = -2^{-1}\zeta(e_3)_{(x_1,x_2,0)}.$$

Thus, we have

$$(\nabla^2_{e_2} e_1)_{(x_1,x_2,0)} = -\zeta^2/[4(1+\zeta)](e_1)_{(x_1,x_2,0)},$$
$$(\nabla^2_{e_2} e_3)_{(x_1,x_2,0)} = -\zeta^2/[4(1+\zeta)](e_3)_{(x_1,x_2,0)}.$$

By the above equations and $g(e_1, e_3)_{(x_1,x_2,0)} = 0$, if we take a curve $\eta_o: s \to (0, s, 0)$, $s \in \mathbf{R}$, then there exist vectors $u$ and $v$ of $T_o(\mathbf{R}^3)$ such that $g(u, v) = 0$, $\| u \| = \| v \|$ and

$$\begin{aligned}(e_1)_{(0,s,0)} &= \cos(\zeta/2\sqrt{1+\zeta})s\,(\eta_0)^0_s u \\ &\quad + \sin(\zeta/2\sqrt{1+\zeta})s\,(\eta_0)^0_s v, \\ (e_3)_{(0,s,0)} &= 1/\sqrt{1+\zeta}\,[\sin(\zeta/2\sqrt{1+\zeta})s\,(\eta_0)^0_s u \\ &\quad - \cos(\zeta/2\sqrt{1+\zeta})s\,(\eta_0)^0_s v].\end{aligned}$$

We note that this is also an example of 3-torus $(T^3, g)$.

The following example is connected with Proposition 4.

**Example 6.** Let $(M, g)$ be a warped product of $S^3$ and $S^1$ defined by

$$\begin{aligned} M = S^3 \times S^1 &= \{(\mathrm{a}, u) \mid \mathrm{a} \in S^3 \text{ and } u = \exp(\sqrt{-1}\,\theta)\} \\ &= \{(a_1, a_2, a_3, a_4, u) \mid a_i \in \mathbf{R} \text{ such that } \textstyle\sum a_i^2 = 1\}, \\ g_{(\mathrm{a},u)} &= \sum_{i=1}^{4} (da_i)^2 + f(\mathrm{a})(d\theta)^2, \end{aligned}$$

where $f(\mathrm{a})$ is a positive function on $S^3$. Let $r(\mathrm{a}) = a_1^2 + a_2^2$. We take $f(\mathrm{a}) = h[r(\mathrm{a})]$. In particular, for a sufficiently small $\varepsilon > 0$, we take

$$(1) \qquad h(r) = \begin{cases} 1-16(r-1/4)+34(r-1/4)^2 & \text{for } |r-1/4| < \varepsilon \\ h(r) > 0 & \text{for } r \in [0,1]. \end{cases}$$

Put

$$\begin{cases} (X)_{(\mathrm{a},u)} = (-a_2\partial/\partial a_1 + a_1\partial/\partial a_2 - a_4\partial/\partial a_3 + a_3\partial/\partial a_4 + \partial/\partial\theta)_{(\mathrm{a},u)}, \\ (Y)_{(\mathrm{a},u)} = (-a_2\partial/\partial a_1 + a_1\partial/\partial a_2 + a_4\partial/\partial a_3 - a_3\partial/\partial a_4)_{(\mathrm{a},u)}. \end{cases}$$

Let $\pi: M \to S^3$ be a projection given by $\pi[(\mathrm{a}, u)] = \mathrm{a}$. For a vector field $\mathrm{w}$ on $S^3$, let $\bar{\mathrm{w}}$ be a vector field on $M$ such that $\pi_*(\bar{\mathrm{w}}) = \mathrm{w}$ and $g(\bar{\mathrm{w}}, \partial/\partial\theta) = 0$. Put $\pi_*(X) = \mathrm{x}$ and $\pi_*(Y) = \mathrm{y}$, then $X = \bar{\mathrm{x}} + \partial/\partial\theta$ and $Y = \bar{\mathrm{y}}$. $X$ and $Y$ are Killing vector fields on $M$ such that $[X, Y] = 0$, and $X$ and $Y$ are linearly independent.

Put $\bar{Y} = Y - [g(Y, X)/\| X \|^2]X$, then the norm $\| \bar{Y} \|$ attains a local minimum at a point $(\mathrm{a}, u)$ such that $r(\mathrm{a}) = 1/4$. Put $N = \{(\mathrm{a}, u) \in M \mid r(\mathrm{a})$

$= 1/4\}$. For a point $p$ of $N$, we have $[g(Y,X)/\|X\|^2] = -1/4$.

Put $Z = Y+(1/4)X$, then an integral curve $\eta_0 : s \to x(s)$, $s \in \mathbf{R}$, of $Z$ through a point $p$ of $N$ is a geodesic in $M$. Then, there exists a system $\{u, v, e\}$ of vectors in $T_p(M)$ such that $\{Z_p, u, v, e\}$ is an orthogonal in $T_p(M)$ and $\|u\| = \|v\|$. Furthermore, if put $c = \sqrt{69}/4$, we have

$$\begin{cases}(X)_{x(s)} = \cos cs(\eta_0)_s^0 u + \sin cs(\eta_0)_s^0 v + (\eta_0)_s^0 e \\ (\eta_0)_s^0 e = (1/23)\{14(\bar{\mathrm{x}} - 4\bar{\mathrm{y}}) + 60Z\}_{x(s)}.\end{cases}$$

We denote by $\bar{\nabla}$ the covariant differentiation with respect to $g$, then we have

$$\begin{cases}K(X \wedge Z)_{x(s)} = c^2(\|X-(\eta_0)_s^0 e\|^2/\|Z\|^2\|X\|^2)[x(s)] > 0 \\ K[(\eta_0)_s^0 e \wedge Z]_{x(s)} = 0,\ K(\bar{\nabla}_Z X \wedge Z)_{x(s)}(= 261/128) > 0.\end{cases}$$

Furthermore, we have that, at a point $(\mathrm{a}, u)$ such that $|r(\mathrm{a})-1/4| < \varepsilon_1$ $(0 < \varepsilon_1 < \varepsilon)$, all sectional curvatures are non-negative, i.e., $K(\bar{\mathrm{x}} \wedge \partial/\partial\theta) = 0$, $K(\bar{\mathrm{y}} \wedge \partial/\partial\theta) = 0$ and $K(\bar{\nabla}_{\bar{\mathrm{x}}}\bar{\mathrm{y}} \wedge \partial/\partial\theta) > 0$.

We shall show the above statement. The vector fields on the standard sphere $S^3$

$$\begin{cases}\mathrm{x} = -a_2\partial/\partial a_1 + a_1\partial/\partial a_2 - a_4\partial/\partial a_3 + a_3\partial/\partial a_4 \\ \mathrm{y} = -a_2\partial/\partial a_1 + a_1\partial/\partial a_2 + a_4\partial/\partial a_3 - a_3\partial/\partial a_4\end{cases}$$

are infinitesimal vector fields given by 1-parameter groups

$$\phi_t = \begin{bmatrix}\exp(\sqrt{-1}\,t) & 0 \\ 0 & \exp(\sqrt{-1}\,t)\end{bmatrix}, \qquad (2)$$
$$\psi_s = \begin{bmatrix}\exp(\sqrt{-1}\,s) & 0 \\ 0 & \exp(-\sqrt{-1}\,s)\end{bmatrix},$$

respectively. (We used the same notations as in Example 2.) Therefore, $[\mathrm{x},\mathrm{y}] = 0$ and every integral curves of $\mathrm{x}$ and $\mathrm{y}$ are geodesics in $S^3$.

At a point $p$ of $N$, the norm $\|\bar{Y}\|$ attains a local minimum. In fact, we have $\|\bar{Y}\|^2 = 1-(2r-1)^2/(1+h(r))$, $(\|\bar{Y}\|^2)'|_{r=1/4} = 0$ and $(\|\bar{Y}\|^2)''|_{r=1/4} = 1/4$. Therefore, the curve $\eta_0$ is a geodesic [see § 3, Prop.6 and Lemma 3.1].

We denote by $\nabla$ the covariant differentiation of $S^3$. Then, we have $(\bar{\nabla}_{\bar{v}}\bar{\mathrm{w}})_{(\mathrm{a},u)} = (\overline{\nabla_v \mathrm{w}})_{(\mathrm{a},u)}$. Furthermore, we denote by $\nabla'$ the covariant differentiation of Euclidian space $\mathbf{R}^4$. Put $e_i = \partial/\partial a_i - a_i\mathrm{a}$, where $\mathrm{a} = \sum_{i=1}^4 a_i\partial/\partial a_i$, then

(3) $$\nabla_{e_i} e_j = T(S^3)\text{-component of } \nabla'_{e_i} e_j = -a_j e_i.$$

Using the above notations, we have

(4) $$\begin{cases} x = -a_2e_1 + a_1e_2 - a_4e_3 + a_3e_4 \\ y = -a_2e_1 + a_1e_2 + a_4e_3 - a_3e_4. \end{cases}$$

*We have, at a point* $(\mathrm{a}, u)$ *such that* $r(a) \neq 0$ *or* $r(\mathrm{a}) \neq 1$,

(5) $$\begin{cases} \bar{\nabla}_{\partial/\partial\theta}(\partial/\partial\theta) = (1/2)h'(\overline{\nabla_x \mathrm{y}}) \\ \bar{\nabla}_{\partial/\partial\theta}(\bar{\mathrm{w}}) = -(h'/2h)g(\mathrm{w}, \nabla_x\mathrm{y})\partial/\partial\theta. \end{cases}$$

*Proof of* (5). Using $[\partial/\partial\theta, \bar{\mathrm{w}}] = 0$, we have

$$\begin{cases} 2g(\bar{\nabla}_{\partial/\partial\theta}(\partial/\partial\theta, \bar{\mathrm{w}}) = -\bar{\mathrm{w}}\cdot g(\partial/\partial\theta, \partial/\partial\theta) = -\mathrm{w}\cdot h \\ 2g(\bar{\nabla}_{\partial/\partial\theta}(\partial/\partial\theta), \partial/\partial\theta) = 0 \\ 2g(\bar{\nabla}_{\partial/\partial\theta}(\bar{\mathrm{w}}), \bar{\mathrm{v}}) = 0 \\ 2g(\bar{\nabla}_{\partial/\partial\theta}(\bar{\mathrm{w}}), \partial/\partial\theta) = \bar{\mathrm{w}}\cdot g(\partial/\partial\theta, \partial/\partial\theta) = \mathrm{w}\cdot h. \end{cases}$$

Since $h$ is a function with respect to $r$, we have $\mathrm{w}\cdot h = 0$ for $\mathrm{w} = \mathrm{x}$ or $\mathrm{y}$, and $\mathrm{w}\cdot h \neq 0$ only for $\mathrm{w}$ such that $g(\mathrm{w}, \mathrm{grad}\, r) \neq 0$. By $g(\nabla_x\mathrm{y}, \mathrm{y}) = g(\nabla_x\mathrm{y}, \mathrm{x}) = 0$, $\nabla_x\mathrm{y}$ and grad $r$ are linearly dependent at each point a such that $r(\mathrm{a}) \neq 0$ or $r(\mathrm{a}) \neq 1$. Therefore, we shall calculate $\nabla_x\mathrm{y}\cdot r$ and $\|\nabla_x\mathrm{y}\|^2$. We have, by (2), (3) and (4),

$$\begin{aligned} \nabla_x\mathrm{y} &= -a_1e_1 - a_2e_2 + a_3e_3 + a_4e_4 + a_2a_1\mathrm{x} - a_1a_2\mathrm{x} - a_4a_3\mathrm{x} + a_3a_4\mathrm{x} \\ &= -a_1e_1 - a_2e_2 + a_3e_3 + a_4e_4. \end{aligned}$$

Thus, we have

(6) $$\|\nabla_x\mathrm{y}\|^2 = 4r(1-r) \text{ and } \nabla_x\mathrm{y}\cdot r = -4r(1-r).$$

Therefore, we have (5).

*We have*

(7) $$\begin{cases} \nabla_x\mathrm{y} = -a_1e_1 - a_2e_2 + a_3e_3 + a_4e_4 \\ \nabla_x^2\mathrm{y} = -\mathrm{y} + (2r-1)\mathrm{x} \\ \nabla_y\nabla_x\mathrm{y} = -\mathrm{x} + (2r-1)\mathrm{y} \\ \nabla_x^3\mathrm{y} = -\nabla_x\mathrm{y}. \end{cases}$$

*Proof of* (7). We have already shown the first equality in Proof of (5).

By (2) and (3), we have

$$\begin{aligned} \nabla_x^2 \mathbf{y} &= \nabla_x[-a_1e_1-a_2e_2+a_3e_3+a_4e_4] \\ &= a_2e_1-a_1e_2-a_4e_3+a_3e_4+(a_1^2+a_2^2-a_3^2-a_4^2)\mathbf{x} \\ &= -\mathbf{y}+(2r-1)\mathbf{x}. \end{aligned}$$

The other equalities in (7) are obtained in the same way.

*We have, at a point p of N,*

$$(8)\qquad \begin{cases} \bar{\nabla}_Z X = -\overline{\nabla_x \mathbf{y}} \\ \bar{\nabla}_Z^2 X = (1/8)[21\bar{\mathbf{x}}+54\bar{\mathbf{y}}]-6Z \\ \bar{\nabla}_Z^3 X = (69/16)(\overline{\nabla_x \mathbf{y}}). \end{cases}$$

*Proof of* (8). By $Z=\bar{\mathbf{y}}+(1/4)\bar{\mathbf{x}}+(1/4)\partial/\partial\theta$, (5) and (7), we have

$$\begin{aligned} \bar{\nabla}_Z X &= (\overline{\nabla_y \mathbf{x}})+(1/4)\bar{\nabla}_{\partial/\partial\theta}(\partial/\partial\theta) \\ &= (\overline{\nabla_y \mathbf{x}})+(1/8)h'(\overline{\nabla_x \mathbf{y}}). \end{aligned}$$

Therefore, we have $\bar{\nabla}_Z X = -\overline{\nabla_x \mathbf{y}}$ at a point $p$ of $N$. And we have, at a point $p$ of $N$,

$$\begin{aligned} \bar{\nabla}_Z^2 X &= -[\overline{\nabla_y \nabla_x \mathbf{y}}+(1/4)\overline{\nabla_x^2 \mathbf{y}}+(1/4)\bar{\nabla}_{\partial/\partial\theta}(\overline{\nabla_x \mathbf{y}})] \\ &= [\bar{\mathbf{x}}+(1/2)\bar{\mathbf{y}}]+(1/4)[\bar{\mathbf{y}}+(1/2)\bar{\mathbf{x}}]-(3/2)\,\partial/\partial\theta \\ &= (1/8)[21\bar{\mathbf{x}}+54\bar{\mathbf{y}}]-6Z. \end{aligned}$$

In the same way, we have the last equality in (8).

By (8), put $c=\sqrt{69}/4$, then there exist vectors $u$ and $v$ of $T_p(M)$ such that $g(u,v)=0$, $\|u\|=\|v\|$. And they satisfy

$$(\bar{\nabla}_Z X)_{x(s)} = c[-\sin cs(\eta_0)_s^0 u+\cos cs(\eta_0)_s^0 v].$$

Therefore, there exists a vector $e$ of $T_p(M)$ such that

$$(X)_{x(s)} = \cos cs(\eta_0)_s^0 u+\sin cs(\eta_0)_s^0 v+(\eta_0)_s^0 e.$$

By (8), we have

$$\begin{aligned} (\eta_0)_s^0 e &= (X)_{x(s)}+(16/69)[(1/8)(21\bar{\mathbf{x}}+54\bar{\mathbf{y}})-6Z]_{x(s)} \\ &= (1/23)[14\bar{\mathbf{x}}-56\bar{\mathbf{y}}+60Z]_{x(s)}. \end{aligned}$$

The equations of sectional curvature stated before are easily shown from (5) and (6).

### 3. Proof of propositions

In this section, we shall prove Lemma and Propositions in § 2. The number of Proposition is same as in § 2, but the number of Lemma and Remark is different from one in § 2. At first, we shall start with the following known Proposition [cf. 4, p.65 Rem.].

**Proposition 6.** *Let $X$ be a Killing vector field on $M$. Then, we have the following facts ;*

(1) *The norm $\|X\|$ of $X$ is constant along each integral curve of $X$.*

(2) *Let $p$ be a point in $M$ at which the square norm $\|X\|^2$ of $X$ is critical and $X_p \neq 0$. Then, the integral curve of $X$ through $p$ is a geodesic in $M$.*

Let $X$ and $Y$ be Killing vector fields on a compact Riemannian manifold $M$ such that $[X, Y] = 0$. We assume that $X$ and $Y$ do not have zero on $M$. Let $\overline{Y}$ be a vector field on $M$ defined by

$$\overline{Y} = Y - [g(Y,X)/\|X\|^2]X.$$

Then, there exists a point $p$ of $M$ at which the square norm $\|\overline{Y}\|^2$ of $\overline{Y}$ is critical. (For a while, we only assume that $\|\overline{Y}\|^2$ is critical at $p$. But, later we add the assumption that $\|\overline{Y}\|^2$ attains a local minimum at $p$.)

**Proposition 1.** *If $\overline{Y}_p = 0$, then the integral curves of $X$ and $Y$ through $p$ coincide (as the image of curves).*

*Proof of Proposition* 1. Because that $X$ and $Y$ are Killing and $[X, Y] = 0$, we have $X \cdot g(X, X) = 0$, $X \cdot g(Y, X) = g([X, Y], X) = 0$ and $X \cdot g(Y, Y) = 2g([X, Y], Y) = 0$. Therefore, by $Y_p = [g(Y,X)/\|X\|^2](p)X_p$, we have $Y = cX$ along the integral curve of $X$ through $p$, where $c = [g(Y,X)/\|X\|^2](p)$.

*q.e.d.*

We assume $X_p \neq 0$, $Y_p \neq 0$ and $\overline{Y}_p \neq 0$. Then, put

$$Z = Y - [g(Y, X)/\|X\|^2](p)X.$$

The vector field $Z$ is also Killing. Let $\{\phi_t\}_{t\in R}$ (resp. $\{\psi_s\}_{s\in R}$) be a 1-parameter group of isometries generated by $X/\|X\|(p)$ (resp. $Z/\|Z\|(p)$).

**Lemma 3.1.** *The norm* $\| Z \|$ (*resp.* $\| \bar{Y} \|$) *of* $Z$ (*resp.* $\bar{Y}$) *is critical at each point* $q$ *of the integral curve of* $Z$ *through* $p$, *and* $g(Z, X) = 0$ *along its curve. Furthermore, if we assume that the norm* $\| \bar{Y} \|$ *attains a local minimum at* $p$, *then* $\| Z \|$ (*resp.* $\| \bar{Y} \|$) *also attains the local minimum at each point* $q$ *of the integral curve of* $Z$ *through* $p$.

*Proof of Lemma* 3.1. Put $c = [g(Y, X)/\| X \|^2](p)$. We have

$$\begin{aligned} Z &= Y - [g(Y, X)/\| X \|^2]X + [g(Y, X)/\| X \|^2 - c]X \\ &= \bar{Y} + [g(Y, X)/\| X \|^2 - c]X. \end{aligned}$$

By $g(\bar{Y}, X) = 0$, we have $\| Z \| \geqq \| \bar{Y} \|$. Furthermore, by $\| Z \|(p) = \| \bar{Y} \|(p)$ and $Z \cdot g(X, X) = Z \cdot g(X, Y) = Z \cdot g(Y, Y) = 0$, we have Lemma. *q.e.d.*

We only consider the case that $M$ has the following Killing vector fields $X$ and $Z$;

$$(1) \qquad \begin{cases} (1) & [X, Z] = 0, \ g(X, Z)(p) = 0. \\ (2) & X_p \neq 0, \ Z_p \neq 0. \\ (3) & \text{The norm } \| Z \| \text{ is critical at the point } p. \end{cases}$$

For the simplicity, we replace $Z/\| Z \|(p)$ (resp. $X/\| X \|(p)$) with $Z$ (resp. $X$). We define an immersion $x$ of $\mathbf{R}^2$ into $M$ by

$$x(t, s) = \phi_t(\psi_s(p)) \quad \text{for } (t, s) \in \mathbf{R}^2.$$

**Proposition 2.** *The immersion* $x$ *is isometric and*

$$(2) \qquad K(X, Z) = g(\nabla_Z X, \nabla_Z X) \quad \textit{on } x(\mathbf{R}^2).$$

*Proof of Proposition* 2. For each $s$ of $\mathbf{R}$ (resp. $t$ of $\mathbf{R}$), we denote by $x(\dot{t}, s)$ (resp. $x(t, \dot{s})$) the vector tangent to the curve $x(t, s)$, $t \in \mathbf{R}$, (resp. $x(t, s)$, $s \in \mathbf{R}$,) at each point $x(t, s)$. Then, we have

$$x(\dot{t}, s) = (\psi_s \phi_t)_* X_p, \ x(t, \dot{s}) = (\phi_t \psi_s)_* Z_p.$$

Thus, we have

$$\begin{aligned} \langle \partial/\partial t, \partial/\partial t \rangle_{\mathbf{R}^2} &= 1 = g(X, X)(p) = g(x(\dot{t}, s), x(\dot{t}, s)) \\ \langle \partial/\partial s, \partial/\partial s \rangle_{\mathbf{R}^2} &= 1 = g(Z, Z)(p) = g(x(t, \dot{s}), x(t, \dot{s})) \\ \langle \partial/\partial t, \partial/\partial s \rangle_{\mathbf{R}^2} &= 0 = g(X, Z)(p) = g(x(\dot{t}, s), x(t, \dot{s})). \end{aligned}$$

This shows that the immersion $x$ is isometric.

We shall prove the equality (2). We consider the following equations on $x(\mathbf{R}^2)$ ;

$$K(X\wedge Z) = g(R(X,Z)Z,X) = g(\nabla_X\nabla_Z Z-\nabla_Z\nabla_X Z,X)$$
$$= -Z\cdot g(\nabla_Z X,X)+g(\nabla_X Z,\nabla_Z X) = g(\nabla_Z X,\nabla_Z X),$$

because of $\nabla_Z Z=0$, $(\phi_t)_*(\nabla_Z Z)=\nabla_Z Z$ and $(\psi_s)_*(\nabla_X Z)=\nabla_X Z$.

*q. e. d.*

**Lemma 3.2.** *Let $\eta_0 : s \to x(s)$, $s\in\mathbf{R}$, be an integral curve of $Z$ through $p$, i.e., $x(s)=\psi_s(p)$. Put $A_s=(\eta_0)^0_s[(\psi_s)_*]_p$, where $(\eta_0)^s_0$ denotes the parallel translation of $T_{x(s)}(M)$ onto $T_p(M)$ along the curve $\eta_0$. Then, there exist an orthonormal basis $\{u_1', v_1',\dots,u_k', v_k', w_1,\dots,w_l\}$ of $T_p(M)$ and non-zero real numbers $\{p_1,\dots,p_k\}$ such that $A_s$ is represented, with respect to this basis of $T_p(M)$, as*

$$A_s=\begin{bmatrix} I_1(s) & & & & & & \\ & I_2(s) & & & & 0 & \\ & & \ddots & & & & \\ & & & I_k(s) & & & \\ & & & & 1 & & \\ & 0 & & & & \ddots & \\ & & & & & & 1 \end{bmatrix}\ (l\ \text{ones}),$$

*where*

$$I_i(s)=\begin{bmatrix}\cos p_i s & -\sin p_i s\\ \sin p_i s & \cos p_i s\end{bmatrix}.$$

*Furthermore, if $X_p=\sum r_i u_i'+\sum s_i v_i'+\sum t_i w_i$, then*

$$X_{x(s)}=\sum[\cos p_i s(\eta_0)^0_s(r_i u_i'+s_i v_i')$$
$$+\sin p_i s(\eta_0)^0_s(-s_i u_i'+r_i v_i')]+\sum t_i(\eta_0)^0_s w_i.$$

**Remark.** In Lemma 3.2, put

(3) $$X_{x(s)}=\sum[\cos p_i s(\eta_0)^0_s u_i+\sin p_i s(\eta_0)^0_s v_i]+(\eta_0)^0_s e,$$

then $\|u_i\|=\|v_i\|$ $(i=1,\cdots,k)$ and $\{Z_p,u_1,v_1,\cdots,u_k,v_k,e\}$ is an orthogonal system in $T_p(M)$.

*Proof of Lemma* 3.2 *and Remark.* Since $\psi_s$ is isometric, we have $A_s \in O(T_p(M))$ for each $s$ of $\mathbf{R}$ and $[(\psi_{s_1})_*]_p(\eta_0)_0^{s_2} = (\eta_0)_{s_1}^{s_1+s_2}[(\psi_{s_1})_*]_{x(s_2)}$. Thus, we have

$$\begin{aligned} A_{s_1} \cdot A_{s_2} &= (\eta_0)_0^{s_1}[(\psi_{s_1})_*]_p(\eta_0)_0^{s_2}[(\psi_{s_2})_*]_p \\ &= (\eta_0)_0^{s_1}(\eta_0)_{s_1}^{s_1+s_2}[(\psi_{s_1})_*]_{x(s_2)}[(\psi_{s_2})]_p \\ &= A_{s_1+s_2}. \end{aligned}$$

Thus, $\{A_s\}_{s\in\mathbf{R}}$ is a 1-parameter group of $SO(T_p(M))$. Therefore, we can take an orthonormal basis of $T_p(M)$ stated in Lemma. Furthermore, by $(\psi_s)_*X = X$, $(\nabla_Z Z)_{x(s)} = 0$ and $g(X, Z)(p) = 0$, we have Lemma and Remark. *q.e.d.*

At the equation (3), put

$$(X_1)_{x(s)} = \sum [\cos p_i s(\eta_0)_s^0 u_i + \sin p_i s(\eta_0)_s^0 v_i]$$

for $s$ of $\mathbf{R}$.

**Proposition 3.** *If* $(X_1)_p = 0$, *then* $(X)_{x(s)} = (\eta_0)_s^0 X_p$ *and* $K(X \wedge Z) = 0$ *on* $\eta_0$.

This Proposition 3 directly follows from Proposition 2.

**Proposition 4.** *We take the point p at which the norm* $\|\bar{Y}\|$ *attains a local minimum. At the equation* (3), *assume* $(X_1)_p \neq 0$ *and* $e \neq 0$. *Then,* $K(Z \wedge (\eta_0)_s^0 e)$ *is non-positive constant along* $\eta_0$.

*Proof of Proposition* 4. We take a curve $y(t)$, $t \in (-\varepsilon, \varepsilon)$ $(\varepsilon > 0)$, such that $y(0) = p$ and $y(\dot{0}) = e$. Fix some $s$ $(> 0)$. For each $t \in (-\varepsilon, \varepsilon)$, let $\tau_t$ be an integral curve of $Z$ through the point $y(t)$, i.e., $\tau_t : r \to \psi_r[y(t)]$, $0 \leqq r \leqq s$. From Lemma 3.2 and Remark, $[(\psi_r)_*]_p e = (\eta_0)_r^0 e$ holds. Therefore, the variation $\{\tau_t\}_{t\in(-\varepsilon,\varepsilon)}$ of the curve $\eta_0|_{[0,s]}$ has the infinitesimal variation vector field $(\eta_0)_r^0 e$ $(0 \leqq r \leqq s)$ along $\eta_0|_{[0,s]}$, where we denote by $\eta_0|_{[0,s]}$ the curve $\eta_0$ restricted, as the parameter space, to the closed interval $[0, s]$. Put $\psi_r[y(t)] = y(t, r)$. Since the norm $\|Z\|$ attains the local minimum at $x(r)$ for each $r$ (by Lemma 3.1), we have

$$L(\eta_0|_{[0,s]}) = \int_0^s \| Z \|[x(r)]dr = \| Z \|(p)s$$
$$\leqq \| Z \|[y(t)]s = \int_0^s \| Z \|[y(t,r)]dr = L(\tau_t).$$

Furthermore, we have

$$[dL(\tau_t)/dt]_{t=0} = \int_0^s g(\nabla_{\dot\eta_0}(\eta_0)_r^0 e, \dot\eta_0)[x(r)]dr = 0.$$

Thus, we have

$$(4) \qquad [d^2L(\tau_t)/dt^2]_{t=0} \geqq 0.$$

We shall calculate the second variation, precisely. Then, by $g(e, Z)(p) = 0$, we have

$$[d^2L(\tau_t)/dt^2]_{t=0}$$
$$(5) \qquad = \int_0^s \{\| \nabla_{\dot\eta_0}(\eta_0)_r^0 e \|^2 - g[R((\eta_0)_r^0 e, \dot\eta_0)\dot\eta_0, (\eta_0)_r^0 e]\}(x(r))\, dr$$
$$+ g(\nabla_{y(\dot 0,s)} y(\dot t, s), \dot\eta_0)[x(s)] - g(\nabla_{y(\dot 0,0)} y(\dot t, 0), \dot\eta_0)[x(0)].$$

In (5), by $(\phi_s)_*[\nabla_{y(\dot 0,0)} y(\dot t, 0)] = \nabla_{y(\dot 0,s)} y(\dot t, s)$, we have

$$g(\nabla_{y(\dot 0,s)} y(\dot t, s), \dot\eta_0)[x(s)] = g(\nabla_{y(\dot 0,0)} y(\dot t, 0), \dot\eta_0)[x(0)].$$

Thus we have, by (4) and (5),

$$0 \leqq [d^2L(\tau_t)/dt^2]_{t=0}$$
$$= -\int_0^s g[R((\eta_0)_r^0 e, \dot\eta_0)\dot\eta_0, (\eta_0)_r^0 e][x(r)]dr$$
$$= -sg(R(e, \dot\eta_0)\dot\eta_0, e)(p),$$

because that $g[R((\eta_0)_r^0 e, \dot\eta_0)\dot\eta_0, (\eta_0)_r^0 e][x(r)]$ is independent of $r$.

*q.e.d.*

**Proposition 5.** *We take the point $p$ at which the norm $\| \bar{Y} \|$ attains a local minimum. At the equation (3), assume $e = 0$. Then, we have that $K(Z \wedge W)$ is negative constant along $\eta_0$, where*

$$W_{x(s)} = \sum p_i^{-1}[\sin p_i s(\eta_0)_s^0 u_i - \cos p_i s(\eta_0)_s^0 v_i].$$

*Proof of Proposition* 5. We take a sufficiently small neighborhood $N$ of $p$. $N^*$ denotes the set of integral curves of $X$ in $N$. Let $\pi: N \to N^*$ denote a mapping which transfers a point $q$ of $N$ to an integral curve of $X$ through $q$. Then, $N^*$ has a Riemannian metric $g^*$ naturally, and then $\pi$ becomes a Riemannian submersion with respect to $g$ and $g^*$. We denote by $\nabla^*$ and $R^*$ the covariant differentiation and curvature of $N^*$ respectively.

For some $s$ of $\mathbf{R}$, we have $\eta_0|_{[0,s]} \subset N$. Then, by $g(\dot{\eta}_0, X) = 0$ and $\nabla_{\dot{\eta}_0}\dot{\eta}_0 = 0$, the curve $\pi(\eta_0|_{[0,s]})$ is a geodesic in $N^*$. Put $\eta_0^* = \pi(\eta_0|_{[0,s]})$.

We take a curve $y(t)$, $-\varepsilon < t < \varepsilon$, such that $y(0) = p$, $y(\dot{0}) = W_p$ and $g(y(\dot{t}), X)[y(t)] = 0$. In fact, we can take such a curve $y(t)$, because of $g(W,X)(p) = 0$. For each $t \in (-\varepsilon, \varepsilon)$, Let $\tau_t$ be an integral curve of $Z$ through the point $y(t)$, i.e., $\tau_t : r \to \psi_r[y(t)]$, $0 \leqq r \leqq s$. Put $\psi_r[y(t)] = y(t,r)$. Then, by Lemma 3.2, the variation $\{\tau_t\}_{t\in(-\varepsilon,\varepsilon)}$ of the curve $\eta_0|_{[0,s]}$ has the infinitesimal variation vector field $W$ along $\eta_0|_{[0,s]}$.

Put $\pi(\tau_t) = \tau_t^*$. Then, $\tau_t^*$ is an integral curve of $\pi_*(Z) = \pi_*(\bar{Y})$. Furthermore, $\bar{Y}$ is the horizontal lift of $\pi_*(\bar{Y})$ with respect to $\pi: N \to N^*$. By Lemma 3.1, the norm $\|\bar{Y}\|$ attains the local minimum at $x(r)$ for each $r$. Therefore, we have

$$\begin{aligned} L(\eta_0^*) &= \int_0^s \|\bar{Y}\|(x(r))dr = \|\bar{Y}\|(p)s \leqq s\,\|\bar{Y}\|(y(t)) \\ &= \int_0^s \|\bar{Y}\|(y(t,r))dr = L(\tau_t^*). \end{aligned}$$

And, by $g(W,X)[x(r)] = 0$, $g(Z,X)[x(r)] = 0$ and $g(\nabla_Z W, Z)[x(r)] = 0$, we have

$$\begin{aligned} &[dL(\tau_t^*)/dt]_{t=0} \\ &= \int_0^s g^*(\nabla^*_{\dot{\eta}_0^*}W^*, \dot{\eta}_0^*)[\pi(x(r))]dr \\ &= \int_0^s g(\nabla_{\dot{\eta}_0}W, \dot{\eta}_0)[x(r)]dr = 0, \end{aligned}$$

where $\pi_*(W) = W^*$ and $\pi_*(\dot{\eta}_0) = \dot{\eta}_0^*$. Thus, we have

$$(6) \qquad [d^2L(\tau_t^*)/dt^2]_{t=0} \geqq 0.$$

We shall calculate the second variation, precisely. Then, by $g^*(W^*, \dot{\eta}_0^*)[\pi(x(r))] = g(W,Z)[x(r)] = 0$, we have

$$(7)\quad [d^2L(\tau_t^*)/dt^2]_{t=0}$$
$$= \int_0^s [\| \nabla^*_{\dot{\eta}_0^*} W^* \|^2 - g^*(R^*(W^*, \dot{\eta}_0^*)\dot{\eta}_0^*, W^*)][\pi(x(r))]dr$$
$$+ g^*(\nabla^*_{y^*(\dot{0},s)} y^*(\dot{t}, s), \dot{\eta}_0^*)[\pi(x(s))]$$
$$- g^*(\nabla^*_{y^*(\dot{0},0)} y^*(\dot{t},0), \dot{\eta}_0^*)[\pi(x(0))],$$

where $\pi_*(y(\dot{t}, r)) = y^*(\dot{t}, r)$. In (7), by $(\phi_s)_*[\nabla_{y(\dot{0},0)} y(\dot{t},0)] = \nabla_{y(\dot{0},s)} y(\dot{t}, s)$, $g(y(\dot{t}, s), X)[y(t, s)] = 0$ and $g(Z, X)[x(r)] = 0$, we have

$$(8)\quad \begin{aligned} &g^*(\nabla^*_{y^*(\dot{0},s)} y^*(\dot{t}, s), \dot{\eta}_0^*)[\pi(x(s))] - g^*(\nabla^*_{y^*(\dot{0},0)} y^*(\dot{t}, s), \dot{\eta}_0^*)[\pi(x(0))] \\ &= g(\nabla_{y(\dot{0},s)} y(\dot{t}, s), \dot{\eta}_0)[x(s)] - g(\nabla_{y(\dot{0},0)} y(\dot{t},0), \dot{\eta}_0)[x(0)] = 0 \end{aligned}$$

and

$$(9)\quad \begin{aligned} &\| \nabla^*_{\dot{\eta}_0^*} W^* \|^2[\pi(x(r))] = \| \text{horizontal part of } \nabla_{\dot{\eta}_0} W \|^2[x(r)] \\ &= \| \text{horizontal part of } X \|^2[x(r)] = 0. \end{aligned}$$

Furthermore, by O'Neill [7], we have

$$(10)\quad \begin{aligned} &g^*(R^*(W^*, \dot{\eta}_0^*)\dot{\eta}_0^*, W^*)[\pi(x(r))] \\ &= g(R(W, \dot{\eta}_0), \dot{\eta}_0, W)[x(r)] + 3 \| \text{vertical part of } \nabla_{\dot{\eta}_0} W \|^2[x(r)] \\ &= g(R(W, \dot{\eta}_0)\dot{\eta}_0, W)[x(r)] + 3 \| X \|^2[x(r)]. \end{aligned}$$

By (6), (7), (8), (9) and (10), we have

$$\begin{aligned} &- \int_0^s [g(R(W, \dot{\eta}_0)\dot{\eta}_0, W) + 3 \| X \|^2][x(r)]dr \\ &= - s[g(R(W, \dot{\eta}_0)\dot{\eta}_0, W) + 3 \| X \|^2](p) \geqq 0. \end{aligned}$$

Thus we have

$$g(R(W, \dot{\eta}_0)\dot{\eta}_0, W)(p) = g(R(W, \dot{\eta}_0)\dot{\eta}_0, W)[x(r)] < 0.$$

*q.e.d.*

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