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ON SOME SERIES ASSOCIATED WITH DISCRETE SUBGROUPS OF U(1, n ; $\mathbb{C}$)

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0. Let F be a fuchsian group acting on the unit disk. An element g_k of F is of the form

$$g_k(z) = \frac{a_k z + \overline{c_k}}{c_k z + \overline{a_k}}, \quad |a_k|^2 - |c_k|^2 = 1.$$

It is well-known about the convergence or divergence of the series $\sum_{g_k \in F} |c_k|^{-t}$ (see [4]). In this paper we show some generalized results on the series associated with discrete subgroups of $U(1, n ; \mathbb{C})$.

1. Let us recall some definitions and notation. Let $V = V^{1,n}(\mathbb{C})$ $(n \geqq 1)$ denote the vector space of $\mathbb{C}^{n+1}$, together with the unitary structure defined by the Hermitian form

$$\Phi(z, w) = -\overline{z_0}w_0 + \overline{z_1}w_1 + \cdots + \overline{z_n}w_n$$

for $z = (z_0, z_1, \ldots, z_n)$ and $w = (w_0, w_1, \ldots, w_n)$. An automorphism g of V, that is a linear bijection of V onto V such that $\Phi(g(z), g(w)) = \Phi(z, w)$ for z, $w \in V$, will be called a unitary transformation. We denote the group of all unitary transformations by $U(1, n ; \mathbb{C})$. Let $V_- = \{z \in V \mid \Phi(z, z) < 0\}$. Obviously V_- is invariant under $U(1, n ; \mathbb{C})$. Let $P(V)$ be the projective space obtained by V.

We define : $H^n(\mathbb{C}) = P(V_-)$. Let $\overline{H^n(\mathbb{C})}$ denote the closure of $H^n(\mathbb{C})$ in projective space $P(V)$. An element g in $U(1, n ; \mathbb{C})$ operates in $P(V)$, leaving $\overline{H^n(\mathbb{C})}$ invariant. Since $H^n(\mathbb{C})$ is identified with the unit ball $B^n(\mathbb{C}) = \{\zeta \mid \|\zeta\|^2 = \sum_{k=1}^{n} |\zeta_k|^2 < 1\}$, we can regard discrete subgroups of $U(1, n ; \mathbb{C})$ as generalized fuchsian groups (see [2]).

2. Let $g_k = (a_{i,j}^{(k)})_{1 \leq i,j \leq n+1}$ be an element in $U(1, n ; \mathbb{C})$. We denote a point of $P^{-1}(0)$ by 0^*. Let d be the derived metric from Ψ (see [2, Proposition 2.4.4]). We easily obtain

Proposition 2.1. $|a_{1,1}^{(k)}| = |\Phi(g_k(0^*), 0^*)| \, |\Phi(0^*, 0^*)|^{-1}$
$= \cosh d(0, g_k(0))$.

For the sake of simplicity and brevity, we denote $2\,|a_{1,1}^{(k)}|$ by $\nu(g_k)$.

Proposition 2.2. *If g and h are elements of $U(1,n;\mathbf{C})$, then*
(1) $\nu(g^{-1}) = \nu(g)$,
(2) $\nu(gh) \leqq \nu(g)\nu(h)$,
(3) $\nu(hgh^{-1}) \leqq [\nu(h)]^2\nu(g) \leqq [\nu(h)]^4\nu(hgh^{-1})$.

Proof. The first is immediate.
(2) Using Proposition 2.1, we have

$$\begin{aligned}\nu(g)\nu(h) &= 2\{\cosh d(0,g(0))\}2\{\cosh d(0,h(0))\}\\ &= 2\{\cosh d(0,g(0))\}2\{\cosh d(g(0),gh(0))\}\\ &\geqq \exp\{d(0,gh(0))\}+\exp\{-d(0,gh(0))\}\\ &= 2\cosh d(0,gh(0))\\ &= \nu(gh).\end{aligned}$$

(3) It follows from (1) and (2) that

$$\begin{aligned}\nu(hgh^{-1}) &\leqq \nu(h)\nu(g)\nu(h^{-1})\\ &= [\nu(h)]^2\nu(g)\\ &= [\nu(h)]^2\nu(h^{-1}hgh^{-1}h)\\ &\leqq [\nu(h)]^4\nu(hgh^{-1}).\end{aligned}$$

3. Unless otherwise stated, we shall always take G to be a discrete subgroup of $U(1,n;\mathbf{C})$. First we give

Definition 3.1 (cf. [3, Theorem 5.1]). For any point $a \in H^n(\mathbf{C})$, G is called of *convergence type or divergence type* according as $\sum_{g\in G}(1-\|g(a)\|)^n$ converges or diverges.

Theorem 3.2. *G is of convergence type or divergence type according as $\sum_{g_k\in G}|a_{1,1}^{(k)}|^{-2n}$ converges or diverges.*

Proof. Noting that $1-\|g_k(0)\|^2 = 1-\sum_{j=2}^{n+1}|a_{j,1}^{(k)}|^2|a_{1,1}^{(k)}|^{-2} = |a_{1,1}^{(k)}|^{-2}$, we see

$$(1/2)(1-\|g_k(0)\|)^{-1} \leqq |a_{1,1}^{(k)}|^2 \leqq (1-\|g_k(0)\|)^{-1}.$$

Therefore we have

$$\sum_{g_k\in G}(1-\|g_k(0)\|)^n \leqq \sum_{g_k\in G}|a_{1,1}^{(k)}|^{-2n} \leqq 2^n\sum_{g_k\in G}(1-\|g_k(0)\|)^n.$$

Thus our proof is complete.

By using (3) in Proposition 2.2, we obtain

Corollary 3.3 (3, Theorem 5.9). *For any element h in $U(1, n; \mathbf{C})$, the conjugate group hGh^{-1} is of the same type as G.*

Next we shall make the estimate of $\sum_{g \in G, \nu(g) < r} [\nu(g)]^{-t}$ as $r \to \infty$. From now on we assume that $G_0 = \{identity\}$.

We now state our results.

Theorem 3.4. *Let $r > 2$ and t any real number. Then*

$$\sum_{g \in G, \nu(g) < r} [\nu(g)]^{-t} = \begin{cases} O(1) \text{ as } r \to \infty \text{ if } t > 2n; \\ O(\log r) \text{ as } r \to \infty \text{ if } t = 2n; \\ O(r^{2n-t}) \text{ as } r \to \infty \text{ if } t < 2n. \end{cases}$$

Theorem 3.5. *Let D_0 be a fundamental polyhedron with respect to 0 for G. If vol (D_0) is finite, then there exist positive numbers m_1 and m_2 such that*

$$m_1 \log r \leqq \sum_{g \in G, \nu(g) < r} [\nu(g)]^{-2n} \leqq m_2 \log r,$$

and, if $t < 2n$, then

$$m_1 r^{2n-t} \leqq \sum_{g \in G, \nu(g) < r} [\nu(g)]^{-t} \leqq m_2 r^{2n-t}.$$

Remark 3.6. When $n = 1$, G is a fuchsian group acting on the unit disk. Noting that the radii of isometric circles are bounded, we see that Theorems 3.4 and 3.5 yield some familiar classical results (see [4]).

For proving the above theorems, we need two lemmas.

Lemma 3.7 (3, Proposition 4.1). *For $0 \leqq r < 1$, the following inequality is satisfied.*

$$n(r, a) \leqq B(1 - r)^{-n},$$

where B is a constant independent of $a \in H^n(\mathbf{C})$.

Lemma 3.8 (3, Proposition 4.4). *Let D_0 be a fundamental polyhedron with respect to 0 for G. Suppose vol $(D_0) < \infty$. Let $a \in D_0$ and $\| a \| < \rho$*

< 1. *Then there exists* r_0 *such that the following inequality is satisfied for* $r_0 \leqq r < 1$.

$$A(1-r)^{-n} \leqq n(r, a) \leqq B(1-r)^{-n},$$

where A *is a constant which depends on* ρ *and* B *is a numerical constant.*

We shall prove Theorems 3.4 and 3.5 in the same manner as in the proof of [1, Theorems 2 and 3].

Proof of Theorems 3.4 *and* 3.5. Let $\chi_0(r) = \#\{g \in G \mid \nu(g) < r\}$. By Lemma 3.7, we have

$$\begin{aligned} \chi_0(r) &= \#\{g \in G \mid \|g(0)\| < \{1-(4/r^2)\}^{1/2}\} \\ &= n(\{1-(4/r^2)\}^{1/2}, 0) \\ &\leqq B(1-\{1-(4/r^2)\}^{1/2})^{-n} \leqq 2^{-n}Br^{2n}. \qquad (1) \end{aligned}$$

For each real number t, we define

$$\chi_t(r) = \sum_{g \in G, \nu(g) < r} [\nu(g)]^{-t}.$$

If $r > 2$, then

$$\chi_t(r) = \int_2^r \frac{d\chi_0(s)}{s^t} = \frac{\chi_0(r)}{r^t} + t \int_2^r \frac{\chi_0(s)}{s^{t+1}} ds. \qquad (2)$$

Using this equation, together with the inequality (1), we obtain Theorem 3.4.

Lemma 3.8 establishes

$$\chi_0(r) \geqq A(1-\{1-(4/r^2)^{1/2}\})^{-n} \geqq 2^{-2n}Ar^{2n}. \qquad (3)$$

By (2) and (3), we complete our proof of Theorem 3.5.

Theorems 3.2 and 3.5 lead to

Corollary 3.9 (3, Theorem 5.4). *If* $vol(D_0) < \infty$, *then* G *is of divergence type.*

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