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ON SUBGROUPS OF CONVERGENCE OR DIVERGENCE TYPE OF U(1, n ; $\mathbb{C}$)

SHIGEYASU KAMIYA

0. Introduction. Let $V = V^{1,n}(\mathbb{C})$ $(n \geqq 1)$ denote the vector space $\mathbb{C}^{n+1}$, together with the unitary structure defined by the Hermitian form

$$\Phi(z, w) = -\bar{z}_0 w_0 + \bar{z}_1 w_1 + \bar{z}_2 w_2 + \cdots + \bar{z}_n w_n$$

for $z = (z_0, z_1, \cdots, z_n)$ and $w = (w_0, w_1, \cdots, w_n)$.
An automorphism g of V, that is a linear bijection of V onto V such that $\Phi(g(z), g(w)) = \Phi(z, w)$ for $z, w \in V$, will be called a unitary transformation. We denote the group of all unitary transformations by $U(1, n; \mathbb{C})$. A unitary transformation operates in $P(V)$, leaving $\overline{H^n(\mathbb{C})}$ invariant. Since $H^n(\mathbb{C})$ is identified with the unit ball $B^n(\mathbb{C})$, discrete subgroups of $U(1, n;$ $\mathbb{C})$ are considered as generalized Fuchsian groups.

In this paper, we shall classify discrete subgroups of $U(1, n; \mathbb{C})$ into convergence type and divergence type as in Fuchsian groups and generalize some results in [3] to them.

1. Preliminaries. Let $V_0 = \{z \in V \mid \Phi(z, z) = 0\}$ and $V_- = \{z \in V \mid \Phi(z, z) = 0\}$. V_0 and V_- are invariant under $U(1, n; \mathbb{C})$. Let $P(V)$ be the projective space obtained from V. This is defined, as usual, by using the equivalent relation in $V - \{0\}$: $u \sim v$ if there exists $\lambda \in \mathbb{C} - \{0\}$ such that $u = v\lambda$. $P(V)$ is the set of equivalence classes, with the quotient topology. Let $P : V - \{0\} \to P(V)$ denote the projection map. We define : $H^n(\mathbb{C}) = P(V_-)$. Let $\overline{H^n(\mathbb{C})}$ denote the closure of $H^n(\mathbb{C})$ in the projective space $P(V)$. An element g in $U(1, n; \mathbb{C})$ operates in $P(V)$, leaving $\overline{H^n(\mathbb{C})}$ invariant. If $z = (z_0, z_1, \cdots, z_n) \in V_-$, then the condition $-|z_0|^2 + \sum_{k=1}^{n} |z_k|^2 < 0$ implies that $z_0 \neq 0$. Therefore we may define a set of coordinates $\zeta = (\zeta_1, \zeta_2, \cdots, \zeta_n)$ in $H^n(\mathbb{C})$ by $\zeta_i(P(z)) = z_i z_0^{-1}$. In this way $H^n(\mathbb{C})$ becomes identified with the unit ball $B = B^n(\mathbb{C}) = \{\zeta = (\zeta_1, \zeta_2, \cdots, \zeta_n) \in \mathbb{C}^n \mid \sum_{k=1}^{n} |\zeta_k|^2 < 1\}$. Next we shall consider the metric in $H^n(\mathbb{C})$. Let $V_{-1} = \{z \in V \mid \Phi(z, z) = -1\}$. Let $T_z(V_{-1})$ be the tangent space. This contains the $\mathbb{C}$-subspace $T_z'(V_{-1}) = \{v \in V \mid \Phi(z, v) = 0\}$. Thus the restriction $P_z' = P_z^* \mid T_z'(V_{-1})$ is a $\mathbb{C}$-linear isomorphism of $T_z'(V_{-1})$ onto

$T_{P(z)}(B)$, where $P_z{}^*: T_z(V_{-1}) \to T_\zeta(B)$. We define the form Ψ in $T_{P(z)}(B)$ by $\Psi(P_z'(v), P_z'(w)) = z_0\Phi(v, w)z_0^{-1}$ which is Hermitian. We can compute this form explicitly, with respect to the standard basis $\{f_1, f_2, \cdots, f_n\}$ in $\mathbf{C}^n$. We have

$$\Psi(f_i, f_j) = \delta_{ij}(1 - \sum_{k=1}^{n} |\zeta_k|^2)^{-1} + \zeta_i\bar{\zeta}_j(1 - \sum_{k=1}^{n} |\zeta_k|^2)^{-2}$$

(c.f. [1], Proposition 2.3.1).

2. The metric δ. We introduce another metric $\delta(a,b)$ for two points a, b in $H^n(\mathbf{C})$ as follows:

$$\delta(a,b) = [1 - \{\Phi(a^*, a^*)\ \Phi(b^*, b^*)\}|\Phi(a^*, b^*)|^{-2}]^{1/2},$$

where $a^* \in P^{-1}(a)$ and $b^* \in P^{-1}(b)$. We see that $\delta(a,b) = \delta(b,a) \geqq 0$ and $\delta(a,c) \leqq \delta(a,b) + \delta(b,c)$, where $a, b, c \in H^n(\mathbf{C})$. Also, if f is an element of $U(1, n; \mathbf{C})$, then $\delta(f(a), f(b)) = \delta(a, b)$. We define $\|a\| = \left\{\sum_{k=1}^{n} |a_k|^2\right\}^{1/2}$, where $a = (a_1, a_2, \cdots, a_n) \in H^n(\mathbf{C})$. Let ρ be a real number satisfying $0 \leqq \rho < 1$. We define

$$C(a,\rho) = \{z \in H^n(\mathbf{C}) \mid \delta(a,z) < \rho\},$$

and then we have the following proposition.

Proposition 2.1.
(i) $C(0,\rho) = \{z \mid \|z\| < \rho\}$.
(ii) *$f(C(a,\rho)) = C(f(a),\rho)$ for any unitary transformation f.*
(iii) *If $\|a\| < r < 1$, then $C(a,\rho)$ is contained in*
$\{z \mid \|z\| < (r+\rho)(1+r\rho)^{-1}\}$.
(iv) $C(a,\rho) \subset \{z \mid \|z - a\| < [\rho^2(1 - \|a\|^2)(1-\rho^2)^{-1}]^{1/2}\}$.

Proof. The first is immediate.
(ii) First we note that $f(C(0,\rho)) = C(f(0),\rho)$. Using Proposition 2.1.2 in [1], we can find $g \in U(1, n; \mathbf{C})$ such that $g(a) = 0$. From this, we obtain

$$g^{-1}(C(0,\rho)) = C(g^{-1}(0),\rho) = C(a,\rho),$$

and therefore

$$f(C(a,\rho)) = fg^{-1}(C(0,\rho)) = C(fg^{-1}(0),\rho) = C(f(a),\rho).$$

(iii) Without loss of generality, we may assume $a = (t, 0, \cdots, 0)$ $(t > 0)$.

Simple computation yields :

$$\{z \mid \delta(a,z) < \rho\} = \{z = (z_1,\cdots,z_n) \mid |(1-t^2\rho^2)^{1/2}z_1 - (1-\rho^2)t(1-t^2\rho^2)^{-1/2}|^2 + (1-t^2)\sum_{j=2}^{n}|z_j|^2 < \rho^2(1-t^2)^2 (1-t^2\rho^2)^{-1}\}$$

It follows that

$$C(a,\rho) \subset \{z \mid \|z\| < (\rho+t)(1+t\rho)^{-1}\} \subset \{z \mid \|z\| < (\rho+r)(1+r\rho)^{-1}\}.$$

(iv) By computation, we obtain the result.

Proposition 2.2. *Let* $\{a_n\}$ *and* $\{b_n\}$ *be sequences in* $H^n(\mathbf{C})$. *Suppose that* $\delta(a_n, b_n) = \rho(constant) < 1$ *and* $\lim_{n\to\infty}\|a_n\| = 1$. *Then* $\lim_{n\to\infty}\|a_n - b_n\| = 0$.

Proof. By (iv) in Proposition 2.1, we see that $\|a_n - b_n\| < \{\rho^2(1-\|a_n\|^2)(1-\rho^2)^{-1}\}^{1/2}$. As $\|a_n\| \to 1$, the value on the right side goes to 0. Therefore we have $\lim_{n\to\infty}\|a_n - b_n\| = 0$.

3. The fundamental polyhedron. Let G be a discrete subgroup of $U(1,n;\mathbf{C})$. Let $a \in H^n(\mathbf{C})$. Suppose the isotropy group $G_a = \{f \in G \mid f(a) = a\} = \{identity\}$. We define the fundamental polyhedron D_a by $\{z \mid \delta(z,a) < \delta(z,f(a))$ for all f in $G\backslash\{identity\}\}$. Obviously we see

$$D_a = \{z \mid d(z,a) < d(z,f(a)) \text{ for all } f \text{ in } G\backslash\{identity\}\},$$

where d is the metric derived from Ψ. Let

$$f_k = \begin{pmatrix} a_{1,1}^{(k)} & \cdots & a_{1,n+1}^{(k)} \\ & \cdots & \\ a_{n+1,1}^{(k)} & \cdots & a_{n+1,n+1}^{(k)} \end{pmatrix}$$

We find that

$$D_0 = \{z \mid \|f_k(z)\| > \|z\| \text{ for all } f_k \in G\backslash\{identity\}\}.$$
$$= \{z = (z_1, z_2, \cdots, z_n) \mid |a_{1,1}^{(k)} + \sum_{j=2}^{n+1} a_{1,j}^{(k)} z_{j-1}| > 1$$

for all $f_k \in G\backslash\{identity\}\}$.

Following the methods of Tsuji [3], we can prove that

(1) No two points in D_a are equivalent under G.
(2) Every point in $H^n(\boldsymbol{C})$ has its equivalent point in $\overline{D_a}$.

4. The counting function n(r, a). Unless otherwise stated, we shall always take G to be a discrete subgroup of $U(1, n; \boldsymbol{C})$ with $G_0 = \{identity\}$. Let a be a point in $H^n(\boldsymbol{C})$. Let $n(r, a)$ be the number of the elements f in G such that $\|f(a)\| < r$. First we prove

Proposition 4.1. *For* $0 \leqq r < 1$, *the following inequality is satisfied.* $n(r, a) \leqq B(1-r)^{-n}$, *where* B *is a constant independent of* a.

For the proof of this proposition, we need two lemmas.

Lemma 4.2. $n(r, a) = \#\{f \in G \mid f(0) \in C(a, r)\}$.

Proof. Let us write $G = \{f_0, f_1, \cdots\}$. Suppose that $\|f_k(a)\| < r$. This means that $\delta(f_k(a), 0) < r$ and so $\delta(a, f_k^{-1}(0)) < r$. Then $f_k^{-1}(0)$ lies in $C(a, r)$. It is similarily seen that $f_k(0) \in C(a, r)$ implies $\|f_k^{-1}(a)\| < r$.

Lemma 4.3.
(i) *The volume element* dV *at* z *in* $H^n(\boldsymbol{C})$ *is*
$K\cdot(1-\|z\|^2)^{-(n+1)}dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge dz_n \wedge d\bar{z}_n$,
where K *is a constant.*
(ii) $\int_{\|z\|<r_1} dV \leqq K_1\cdot(1-r_1)^{-n}$, *where* K_1 *is a constant.*

Proof. (i) Consider det $\{\Psi(f_i, f_j)\}$ to obtain the result. (ii) We see that

$$\int_{\|z\|<r_1} dV \leqq constant \int_0^{r_1} (1-r^2)^{-(n+1)} r dr \leqq K_1(1-r_1)^{-n}.$$

Proof of Proposition 4.1. We first note that G is discontinuous in $H^n(\boldsymbol{C})$. From this fact, we can choose $s > 0$ so small that $f_i(C(0, s)) \cap f_j(C(0, s)) = \emptyset$ for $f_i \neq f_j$. Suppose $\delta(f(0), a) < r$. By Proposition 2.1, we have $f(C(0, s)) = C(f(0), s) \subset C(a, r_1)$, where $r_1 = (r+s)(1+rs)^{-1}$. By Lemma 4.2, we see that the number of images $f_i(C(0, s))$ in $C(a, r_1)$ is $n(r, a)$. Therefore it follows that

$$n(r, a)\ vol(C(0, s)) \leqq vol(C(a, r_1)),$$

where $vol(\cdot)$ denotes the volume. By Proposition 2.1.2 in [1], there exists

$g \in U(1, n; \mathbf{C})$ such that $g(a) = 0$, so we see that $g(C(a, r_1)) = C(g(a), r_1) = C(0, r_1)$. Hence we obtain the inequality

$$n(r, a) \leqq vol(C(0, r_1))(vol(C(0, s))^{-1}.$$

It follows from $r_1 = (r+s)(1+rs)^{-1}$ that $(1-r_1)^{-n} \leqq 2(1-s)^{-n}(1-r)^{-n}$. Using this inequality and Lemma 4.3, we have

$$n(r, a) \leqq constant \cdot (1-s)^{-n}(1-r)^{-n}(vol(C(0, s))^{-1}.$$

The quantity $(1-s)^{-n}(vol(C(0, s))^{-1}$ depends on G. Thus we have the desired inequality.

In the same manner as in Theorem XI. 10 of [3], we obtain

Proposition 4.4. *Suppose $vol(D_0) < \infty$. Let $a \in D_0$ and $\|a\| < \rho < 1$. There exists r_0 such that the following inequality is satisfied for $r_0 \leqq r < 1$.*

$$A(1-r)^{-n} \leqq n(r, a) \leqq B(1-r)^{-n},$$

where A is a constant, which depends on ρ and B is a numerical constant.

5. Convergence type or divergence type. In this section we shall classify discrete subgroups of $U(1, n; \mathbf{C})$ into convergence type and divergence type and discuss their properties.

Theorem 5.1. *Let us write $G = \{f_0, f_1, \cdots\}$. Then either*

(i) $\sum_{f_k \in G} (1 - \|f_k(a)\|)^n < \infty$ *for each $a \in H^n(\mathbf{C})$, or*

(ii) $\sum_{f_k \in G} (1 - \|f_k(a)\|)^n = \infty$ *for each $a \in H^n(\mathbf{C})$.*

Proof. Let $0^* = (\lambda, 0, \cdots, 0)$ and $a^* = (\alpha_1, \alpha_2, \cdots, \alpha_{n+1})$ in V_- such that $P(0^*) = 0$ and $P(a^*) = a$, respectively.
Let

$$f_k = \begin{pmatrix} a_{1,1}^{(k)} & \cdots & a_{1,n+1}^{(k)} \\ & \cdots & \\ a_{1,n+1}^{(k)} & \cdots & a_{n+1,n+1}^{(k)} \end{pmatrix}.$$

We have

$$1 - \|a\|^2 = \{\Phi(f_k(a^*), f_k(a^*))\, \Phi(f_k(0^*), f_k(0^*))\}\, |\Phi(f_k(a^*), f_k(0^*))|^{-2}$$

$$= (1-\|f_k(a)\|^2)(1-\|f_k(0)\|^2)\left|1-\sum_{m=2}^{n+1}\left(\sum_{j=1}^{n+1}\overline{a_{m,j}^{(k)}a_j}a_{m,1}^{(k)}\right)\left(\sum_{j=1}^{n+1}\overline{a_{1,j}^{(k)}a_j}a_{1,1}^{(k)}\right)^{-1}\right|^{-2}$$

$$\leqq (1+\|f_k(a)\|)(1-\|f_k(a)\|)(1+\|f_k(0)\|)(1-\|f_k(0)\|)\left(1-\left|\sum_{m=2}^{n+1}\sum_{j=1}^{n+1}\overline{a_{m,j}^{(k)}a_j}a_{m,1}^{(k)}\left(\sum_{j=1}^{n+1}\overline{a_{i,j}^{(k)}a_j}a_{1,1}^{(k)}\right)^{-1}\right|\right)^{-2}.$$

Noting that $1+\|f_k(a)\|<2$ and $1+\|f_k(0)\|<2$, we obtain the next inequality :

$$1-\|a\|^2 \leqq 4(1-\|f_k(a)\|)(1-\|f_k(a)\|)(1-\|f_k(a)\|\|f_k(0)\|)^{-2}$$
$$\leqq \begin{cases}4(1-\|f_k(a)\|)(1-\|f_k(0)\|)^{-1}\\ 4(1-\|f_k(0)\|)(1-\|f_k(a)\|)^{-1}.\end{cases}$$

Therefore, we see that

$$(1/4)^n(1-\|a\|^2)^n(1-\|f_k(0)\|)^n$$
$$\leqq (1-\|f_k(a)\|)^n$$
$$\leqq 4^n(1-\|f_k(0)\|)^n(1-\|a\|^2)^{-n}.$$

Hence it follows that if $\sum_{f_k\in G}(1-\|f_k(0)\|)^n<\infty$, then $\sum_{f_k\in G}(1-\|f_k(a)\|)^n<\infty$ and that if $\sum_{f_k\in G}(1-\|f_k(0)\|)^n=\infty$, then $\sum_{f_k\in G}(1-\|f_k(a)\|)^n=\infty$. Thus our proof is completed.

Definition. G is called of *convergence type*, or *divergence type* according to the case (i) or (ii).

Next we shall show that the power n is the best number for the classification of discrete subgroups of $U(1,n;\mathbb{C})$.

Theorem 5.2. *If* $\varepsilon>0$, *then* $\sum_{f_k\in G}(1-\|f_k(a)\|)^{n+\varepsilon}<\infty$.

Proof. Using Proposition 4.1, we have

$$\sum_{\|f_k(a)\|<r}(1-\|f_k(a)\|)^{n+\varepsilon}=\int_0^r(1-t)^{n+\varepsilon}dn(t,a)$$

$$=(1-r)^{n+\varepsilon}n(r,a)-n(0,a)+(n+\varepsilon)\int_0^r(1-t)^{n+\varepsilon-1}n(t,a)dt$$

$$\leqq B(1-r)^{\varepsilon}+(n+\varepsilon)B\int_0^r(1-t)^{\varepsilon-1}dt$$

Therefore, if the condition is satisfied, then $\sum_{f_k \in G} (1 - \| f_k(a) \|)^{n+\varepsilon}$ is convergent as $r \to 1$.

Theorem 5.3. *The following three are equivalent.*

(i) *G is of divergence type.*

(ii) $\sum_{f_k \in G} (1 - \| f_k(a) \|^2)^n (1 - \| a \|^2)^{-n} = \infty$ *for all* $a \in H^n(\mathbf{C})$.

(iii) $\int_0^1 (1-t)^{n-1} n(t,a)\,dt = \infty$.

Proof. First we shall prove that (i) and (ii) are equivalent. Noting that $\| f_k(a) \| < 1$, we obtain the following inequalities :

$$(1 - \| f_k(a) \|^2)^n (1 - \| a \|^2)^{-n} = (1 - \| f_k(a) \|)^n (1 + \| f_k(a) \|)^n (1 - \| a \|)^{-n} (1 + \| a \|)^{-n}$$

$$\begin{cases} \leqq 2^n (1 - \| f_k(a) \|)^n (1 - \| a \|)^{-n} & (1) \\ \geqq (1/2)^n (1 - \| f_k(a) \|)^n (1 - \| a \|)^{-n}. & (2) \end{cases}$$

Considering (1), we see that (i) implies (ii). The inequality (2) shows that (ii) yields (i).

Next we shall show that (i) and (iii) are equivalent. We see that

$$\sum_{\| f_k(a) \| < r} (1 - \| f_k(a) \|)^n = \int_0^r (1-t)^n dn(t,a)$$

$$= (1-r)^n n(r,a) - n(0,a) + \int_0^r n(1-t)^{n-1} n(t,a)\,dt$$

It follows from Proposition 4.1 that $(1-r)^n n(r,a) - n(0,a)$ is convergent as $r \to 1$. Therefore we obtain the stated conclusion.

Theorem 5.4. *If $vol(D_0) < \infty$, then G is of divergence type.*

Proof. Let a be a point in D_0 and $\| a \| < \rho < 1$. Using Proposition 4.4, we see that

$$\int_0^1 (1-t)^{n-1} n(t,a)\,dt \geqq \int_0^1 A(1-t)^{-1} dt = \infty.$$

It follows from Theorem 5.3 that G is of divergence type.

Next we consider the case where G is of convergence type.

Theorem 5.5. *The following* (i) *and* (ii) *are equivalent.*

(i) *G is of convergence type.*

(ii) $\sum_{f_k \in G} |\log \delta(f_k(a), z)^{-1}|^n < \infty$ *for some* z *in* $H^n(\mathbf{C})$.

Furtheremore if the above are satisfied, the series in (ii) *is uniformly convergent in any compact subset of* $H^n(\mathbf{C})$.

For the proof, we need two lemmas.

Lemma 5.6. *Let* $z^* = (z_0, z_1, \ldots, z_n)$ *and* $a^* = (a_0, a_1, \ldots, a_n)$ *be in* V_-. *Then*

$$\begin{aligned}(1 - \|P(a^*)\|\|P(z^*)\|)^2 &\leqq |a_0|^{-2}|z_0|^{-2}|\Phi(a^*, z^*)|^2 \\ &\leqq (1 + \|P(a^*)\|\|P(z^*)\|)^2 \\ &\leqq 4.\end{aligned}$$

Proof. Since $|\Phi(a^*, z^*)|^2 = |a_0|^2|z_0|^2|-1 + \sum_{j=1} a_j z_j z_0^{-1} a_0^{-1}|^2$, we have

$$\begin{aligned}(1 - \sum_{j=1}^{n} |a_j||z_j||z_0|^{-1}|a_0|^{-1})^2 &\leqq |a_0|^{-2}|z_0|^{-2}|\Phi(a^*, z^*)|^2 \\ &\leqq (1 + \sum_{j=1}^{n} |a_j||z_j||z_0|^{-1}|a_0|^{-1})^2.\end{aligned}$$

Using Schwartz's inequality, we obtain

$$\begin{aligned}(1 - \|P(z^*)\|\|P(a^*)\|)^2 &= \left\{1 - \left(\sum_{j=1}^{n} |a_j a_0^{-1}|^2\right)^{1/2}\left(\sum_{j=1}^{n} |z_j z_0^{-1}|^2\right)^{1/2}\right\}^2 \\ &\leqq |a_0|^{-1}|z_0|^{-1}|\Phi(a^*, z^*)|^2 \\ &\leqq \left\{1 + \left(\sum_{j=1}^{n} |a_j a_0^{-1}|^2\right)^{1/2}\left(\sum_{j=1}^{n} |z_j z_0^{-1}|\right)^{1/2}\right\}^2 \\ &= (1 + \|P(z^*)\|\|P(a^*)\|)^2.\end{aligned}$$

Since $\|P(z^*)\| < 1$ and $\|P(a^*)\| < 1$, we obtain the result.

Lemma 5.7. *If* a^* *and* z^* *are in* V_-, *then*

$$\begin{aligned}&(1/2)\ \Phi(a^*, a^*)\Phi(z^*, z^*)|\Phi(a^*, z^*)|^{-2} \leqq \log[\delta(P(a^*), P(z^*))]^{-1} \\ &\leqq (1/2)\Phi(a^*, a^*)\Phi(z^*, z^*)\{|\Phi(a^*, z^*)|^2 - \Phi(z^*, a^*)\Phi(z^*, z^*)\}^{-1}.\end{aligned}$$

Proof. Since $\log(1+x) \leqq x (x \geqq 0)$ and $\log(1-x)^{-1} \geqq x (0 \leqq x \leqq 1)$, we have

$$\begin{aligned}&\log\{\delta(P(a^*), P(z^*))\}^{-2} = \log[1 + \{\delta(P(a^*), P(z^*))\}^{-2} - 1] \\ &\leqq \{\delta(P(a^*), P(z^*))\}^{-2} - 1\end{aligned}$$

$$= \Phi(a^*, a^*)\Phi(z^*, z^*)\{|\Phi(a^*, z^*)|^2 - \Phi(a^*, a^*)\Phi(z^*, z^*)\}^{-1}, \text{ and}$$
$$\log\{\delta(P(a^*), P(z^*))\}^{-2} = \log[1 - \{1 - (\delta(P(a^*), P(z^*))^2\}]^{-1}$$
$$\geqq 1 - \{\delta(P(a^*), P(z^*))\}^2$$
$$= \Phi(a^*, a^*)\Phi(z^*, z^*)|\Phi(a^*, z^*)|^{-2}.$$

Proof of Theorem 5.5. By Lemma 5.6, we have

$$(1/8)(1 - \|a\|)(1 - \|z\|^2) \leqq (1/2)\Phi(a^*, a^*)\Phi(z^*, z^*)|\Phi(a^*, z^*)|^{-2}.$$

Using Lemma 5.7, we obtain

$$\sum_{f_k \in G} 8^{-n}(1 - \|f_k(a)\|)^n(1 - \|z\|^2)^n \leqq \sum_{f_k \in G}[\log\{\delta(f_k(a), z)\}^{-1}]^n.$$

If (ii) is satisfied, then the series on the left side is convergent. Thus G is of convergence type. Next we shall show that (i) implies (ii). By Lemmas 5.6 and 5.7, we see that

$$(1/2)\Phi(a^*, a^*)\Phi(z^*, z^*)\{|\Phi(a^*, z^*)|^2 - \Phi(a^*, a^*)\Phi(z^*, z^*)\}^{-1}$$
$$\leqq (1/2)(1 - \|a\|^2)(1 - \|z\|^2)(\|a\| - \|z\|)^{-2}.$$

Now we assume that $\|z\| < r_1 < 1$. Since G is discontinuous in $H^n(\mathbf{C})$, there exists an integer N such that $\|f_k(a)\| > r$ for $n > N$. So we obtain

$$\sum_{f_k \in G} 2^{-n}(1 - \|z\|^2)^n(1 - \|f_k(a)\|^2)^n(\|f_k(a)\| - \|z\|)^{-2n}$$
$$\leqq \sum_{f_k \in G}(1 - \|f_k(a)\|)^n(r_1 - r)^{-2n}.$$

If G is of convergence type, then the series on the right side is convergent. Thus it is seen that $\sum_{f_k \in G}[\log\{\delta(f_k(a), z)\}^{-1}]^n$ is uniformly convergent. So the proof of Theorem 5.5 is complete.

If G is of convergence type, we can denote $\sum_{f_k \in G}[\log\{\delta(f_k(a), z)\}^{-1}]^n$ by $g_a(z)$. Since $\delta(a, b)$ is $U(1, n; \mathbf{C})$-invariant, we have

$$g_a(h(z)) = \sum_{f_k \in G}[\log\{\delta(f_k(a), h(z)\}^{-1}]^n$$
$$= \sum_{f_k \in G}[\log\{\delta(h^{-1}f_k(a), z)\}^{-1}]^n$$

for any h in G. Set $h^{-1}f_k = h_k$. We see that

$$g_a(h(z)) = \sum_{h_k \in G}[\log\{\delta(h_k(a), z)\}^{-1}]^n = g_a(z)$$

for any h in G. So we have proved

Theorem 5.8. *If G is of convergence type, then $g_a(z)$ is G-invariant.*

Theorem 5.9. *Let G be a discrete subgroup of $U(1,n;\mathbf{C})$. Then G and the conjugate group fGf^{-1} are of the same type for $f\in U(1,n;\mathbf{C})$.*

Proof. Note that $\delta(a,b)$ is $U(1,n;\mathbf{C})$-invariant. Set $b=f(a)$ and $w=f(z)$. Then we obtain

$$\sum_{f_k\in G}[\log|\delta(ff_k(a),f(z))|^{-1}]^n=\sum_{f_k\in G}[\log|\delta(ff_kf^{-1}(b),w)|^{-1}]^n.$$

Thus our proof is complete.

Let σ denote the rotation-invariant positive Borel measure on $\partial H^n(\mathbf{C})$ for which $\sigma(\partial H^n(\mathbf{C}))=1$. We shall show a sufficient condition for G to be of convergence type.

Theorem 5.10. *Let E be the subset with positive measure in $\partial H^n(\mathbf{C})$. If $g(E)\cap h(E)=\emptyset$ for any different elements g and h in G, then G is of convergence type.*

Before proving Theorem 5.10, we give the definition of Poisson kernel and discuss its properties. Let z and ζ be in $H^n(\mathbf{C})$ and $\partial H^n(\mathbf{C})$, respectively. We define *Poisson kernel* as follows:

$$P(z,\zeta)=\{|\zeta_0^*|^2|\Phi(z^*,z^*)||\Phi(z^*,\zeta^*)|^{-2}\}^n,$$

where $z^*=(z_0^*,z_1^*,\cdots,z_n^*)\in P^{-1}(z)$ and $\zeta^*=(\zeta_0^*,\zeta_1^*,\cdots,\zeta_n^*)\in P^{-1}(\zeta)$. It is easy to show that the above definition is well-defined. First we show

Proposition 5.11. *Let z be a point in $H^n(\mathbf{C})$. Let ζ and η be in $\partial H^n(\mathbf{C})$. Let g be an element in $U(1,n;\mathbf{C})$. We have the following properties.*

(1) $P(g(z),g(\zeta))=|(g(\zeta^*))_0|^{2n}|\zeta_0^*|^{-2n}P(z,\zeta)$.

(2) $P(g(z),\zeta)=P(z,g^{-1}(\zeta))P(g(0),\zeta)$.

(3) $P(k\eta,\zeta)=P(k\zeta,\eta)$ *for* $0\leqq k<1$.

(4) $\displaystyle\int_{\partial H^n(\mathbf{C})}P(k\eta,\zeta)d\sigma(\eta)=\int_{\partial H^n(\mathbf{C})}P(k\zeta,\eta)d\sigma(\eta)=1$ *for* $0\leqq k<1$.

(5) $\displaystyle\int_{\partial H^n(\mathbf{C})}P(g^{-1}(0),\zeta)d\sigma(\zeta)=\int_{\partial H^n(\mathbf{C})}|\zeta_0^*|^{2n}|(g(\zeta^*))_0|^{-2n}d\sigma(\zeta)$

$\displaystyle=\int_{\partial H^n(\mathbf{C})}d\sigma.$

Proof. Noting that

$$\begin{aligned}&|\Phi(g(z))^*,(g(\zeta))^*)|\\&=|(g(z))_0^*|\,|(g(z^*))_0|^{-1}|(g(\zeta))_0^*|\,|(g(\zeta^*))_0|^{-1}|\Phi(g(z^*),g(\zeta^*))|,\end{aligned}$$

we easily obtain (1), (2) and (3).

(4) The first equality follows from (2). Set $w=k\eta$. By the Cauchy Formula, we obtain

$$\begin{aligned}\int_{\partial H^n(\mathbb{C})} P(w,\zeta)d\sigma(\zeta)&=\int_{\partial H^n(\mathbb{C})}(|\zeta_0^*|^2|\Phi(w^*,w^*)|\,|\Phi(w^*,\zeta^*)|^{-2})^n d\sigma(\zeta)\\&=\int_{\partial H^n(\mathbb{C})}\{-w_0^*\overline{\zeta_0^*}\Phi(\zeta^*,w^*)^{-1}\}^n\{\zeta_0^*|\Phi(w^*,w^*)|(-w_0^*\Phi(w^*,\zeta^*))^{-1}\}^n d\sigma(\zeta)\\&=1.\end{aligned}$$

(5) It is easy to show that $P(g^{-1}(0),\zeta)=\{|\zeta_0^*|\,|(g(\zeta^*))_0|^{-1}\}^{2n}$. Using (2) and (4), we have

$$\begin{aligned}&\int_{\partial H^n(\mathbb{C})} P(g^{-1}(k\eta),\zeta)d\sigma(\zeta)\\&=\int_{\partial H^n(\mathbb{C})} P(k\eta,g(\zeta))P(g^{-1}(0),\zeta)d\sigma(\zeta)\\&=P(g^{-1}(0),\zeta)\int_{\partial H^n(\mathbb{C})} P(k\eta,g(\zeta))d\sigma(\eta)\\&=P(g^{-1}(0),\zeta).\end{aligned}$$

It follows from (4) that

$$\begin{aligned}&\int_{\partial H^n(\mathbb{C})} P(g^{-1}(0),\zeta)d\sigma(\zeta)\\&=\int_{\partial H^n(\mathbb{C})}\left\{\int_{\partial H^n(\mathbb{C})} P(g^{-1}(k\eta),\zeta)d\sigma(\zeta)\right\}d\sigma(\eta)\\&=\int_{\partial H^n(\mathbb{C})} d\sigma.\end{aligned}$$

Lemma 5.12. $P(z,\zeta)\leqq\{(1+\|z\|)(1-\|z\|)^{-1}\}^n\leqq 2^n(1-\|z\|)^{-n}$.

Proof. First we note that $|\Phi(z^*,\zeta^*)|^2\geqq|z_0^*|\,|\zeta_0^*|(1-\|z\|)^2$ for $z^*=(z_0^*,z_1^*,\cdots,z_n^*)$ and $\zeta^*=(\zeta_0^*,\zeta_1^*,\cdots,\zeta_n^*)$. Using the above fact, we easily see that

$$\begin{aligned} P(z,\zeta) &\leqq \{(1-\|z\|^2)(1-\|z\|)^{-2}\}^n \\ &\leqq \{(1+\|z\|)(1-\|z\|)^{-1}\}^n \\ &\leqq 2^n(1-\|z\|)^{-n}. \end{aligned}$$

Now we are ready to prove our theorem.

Proof of Theorem 5.10. Put $u(z) = \int_E P(z,\zeta)d\sigma(\zeta)$. We have $u(0) = \int_E P(0,\zeta)d\sigma(\zeta) = \sigma(E)$. Using (1) and (5) in Proposition 5.11 and Lemma 5.12, we have

$$\begin{aligned} u(0) &= \int_E P(0,\zeta)d\sigma(\zeta) \\ &= \int_E \{|\zeta_0^*||(g(\zeta^*))_0|^{-1}\}^{2n} P(g(0),g(\zeta))d\sigma(\zeta). \\ &\leqq 2^n(1-\|g(0)\|)^{-n}\int_E \{|\zeta_0^*||(g(\zeta^*))_0|^{-1}\}^{2n} d\sigma(\zeta) \\ &= 2^n(1-\|g(0)\|)^{-n}\int_{g(E)} \{|\zeta_0^*||(g(\zeta^*))_0|^{-1}\}^{2n} \\ &\quad \{|(g(\zeta^*))_0||\zeta_0^*|^{-1}\}^{2n} d\sigma(\eta) \\ &= 2^n\sigma(g(E))(1-\|g(0)\|)^{-n}. \end{aligned}$$

It follows from the above fact that

$$\sigma(E) \leqq 2^n\sigma(g(E))(1-\|g(0)\|)^{-n}.$$

Since $g(E) \cap h(E) = \emptyset$, then we have

$$\begin{aligned} \sum_{g\in G}(1-\|g(0)\|)^n &\leqq 2^n(\sigma(E))^{-1}\sum_{g\in G}\sigma(g(E)) \\ &= 2^n(\sigma(E))^{-1}\sigma(\bigcup_{g\in G} g(E)) \\ &\leqq 2^n(\sigma(E))^{-1}\sigma(\partial H^n(\mathbf{C})) < \infty. \end{aligned}$$

Thus our theorem is completely proved.

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DEPARTMENT OF MECHANICAL SCIENCE
OKAYAMA UNIVERSITY OF SCIENCE

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