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ON NON-COMMUTATIVE GENERALIZED P.P. RINGS

MASAYUKI ÔHORI

As a natural generalization of commutative p.p. rings, A. G. Naoum [6] and later Y. Hirano [4], independently, introduced the concept of commutative generalized p.p. rings (or π-Baer rings in Naoum's terminology) and obtained characterizations of such rings. In the present paper we study (non-commutative) generalized p.p. rings and give, among others, extensions of their results to non-commutative rings. First, reduced generalized right p.p. rings are shown to be p.p. rings (Proposition 1) and basic properties of generalized right p.p. rings corresponding to those of right p.p. rings are proved (Propositions 2, 3, 4). Second, characterizations of generalized p.p. rings with additional conditions are given (Theorems 2, 4). Rings, which are finite direct sum of rings whose classical right quotient rings are local rings with nil Jacobson radicals, are also considered (Theorem 3).

Throughout this paper the word "ring" will mean "non-zero associative ring with identity element" and subrings of a ring R are required to have the same identity element as R. For any ring R, we denote the prime radical and the Jacobson radical of R by $P(R)$ and $J(R)$ respectively. $E(R)$ denotes the set of all idempotents of R. A ring R is called *normal* if every idempotent of R is central. For any non-empty subset X of a ring R, we denote the right annihilator of X in R by $r_R(X)$ or $r(X)$. Similarly the left annihilator of X in R is denoted by $l_R(X)$ or $l(X)$. When $r_R(X) = l_R(X)$, we write it $\mathrm{ann}_R(X)$. A right ideal of the form $r_R(X)$ for some non-empty subset X of R, is called a *right annihilator*. A *left annihilator* is defined similarly. A *Baer ring* is a ring in which every right annihilator (or equivalently, every left annihilator) is generated by an idempotent. A *right* (resp. *left*) *p.p. ring* is a ring in which every principal right (resp. left) ideal is projective. Note that for any element x of a ring R, xR is projective if and only if $r(x) = eR$ for some idempotent e of R. A ring R is called a *generalized right* (resp. *left*) *p.p. ring* if for any element x of R, x^nR (resp. Rx^n) is projective for some positive integer n (depending on x). A ring which is both generalized right and left p.p. is said to be a *generalized p.p. ring*. Let n be a positive integer. A ring R is called an *n-generalized right p.p. ring* if x^nR is projective for every element x of R. Let R be a ring and let x be an element of R. We call x *π-regular* if there exists an integer n and an element y of R such that $x^n y x^n = x^n$. Thus R is a *π-regular ring* provided every element of R is π-regular.

We begin with some elementary properties of generalized right p.p. rings.

Proposition 1. *Let R be a reduced, generalized right p.p. ring. Then R is a p.p. ring, that is, R is left p.p. as well as right p.p.*

Proof. For any element $x \in R$, there exists a positive integer n and $e \in E(R)$ such that $r(x^n) = eR$. Since R is reduced and hence normal, we have $(xe)^n = x^n e = 0$, so that $xe = 0$. Hence $r(x) = eR$. By hypothesis we also have $l(x) = r(x)$. Therefore R is a p.p. ring.

As is well known, every right p.p. ring is right non-singular (see, e.g. [1, Lemma 8.3, p. 111]). As to generalized right p.p. rings we have

Proposition 2. *If R is a generalized right p.p. ring, then the right singular ideal $Z_r(R)$ of R is nil.*

Proof. Let x be an arbitrary element of $Z_r(R)$. By hypothesis there exists a positive integer n and $e \in E(R)$ such that $r(x^n) = eR$. If $x^n \neq 0$, then $e \neq 1$, which contradicts $x^n \in Z_r(R)$.

We need the following lemma.

Lemma 1 ([2, Lemma 1.1, p. 1]). *Let A be a non-zero right ideal of a ring R. If there exists a positive integer n such that $a^n = 0$ for every element a of A, then R has a non-zero nilpotent ideal.*

A ring R is said to have *enough idempotents* if the identity element of R can be written as the sum of a finite number of orthogonal primitive idempotents of R.

Proposition 3 (cf. [1, Lemma 8.6, p. 113]). *If R is a semiprime n-generalized right p.p. ring with enough idempotents, then R has no non-zero nil right (and left) ideals. In particular, R is right non-singular.*

Proof. If $n = 1$, then by [1, Lemma 8.6] the assertion is obvious. So suppose that $n > 1$. Let K be a nil right ideal of R. By hypothesis there exist orthogonal primitive idempotents $e_1, \cdots, e_t$ such that $e_1 + \cdots + e_t = 1$.

Let k be an arbitrary element of K. Then ke_i is nilpotent for each i. We contend that $(ke_i)^n = 0$ for every i. Suppose that $(ke_i)^n \neq 0$ for some i and let m be the smallest positive integer such that $(ke_i)^m = 0$. We consider the R-homomorphism $f\colon e_iR \to (ke_i)^nR$ defined by $f(e_ir) = (ke_i)^nr$ for all $r \in R$. Obviously f is surjective and $(ke_i)^nR$ is projective by hypothesis, so that $\mathrm{Ker}\, f$ is a direct summand of e_iR. On the other hand e_iR is indecomposable, because e_i is primitive. Therefore $\mathrm{Ker}\, f = 0$ and so $e_i(ke_i)^{m-n} = 0$. Hence we have $(ke_i)^{m-(n-1)} = 0$, which contradicts the choice of m. From what we have just proved, it follows that $x^{n+1} = 0$ for all $x \in e_iK$ $(i = 1, \cdots, t)$. Then by Lemma 1 we have $e_iK = 0$ $(i = 1, \cdots, t)$, so that $K = (e_1 + \cdots + e_t)K = 0$. Now let L be a nil left ideal of R and let x be an element of L. Noting that xR is a nil right ideal of R, we have $xR = 0$, that is, $x = 0$. Thus $L = 0$ and the proof is complete.

A slight modification of the proof of [8, Theorem 1] yields the next proposition.

Proposition 4. *Let R be a generalized right p.p. ring in which there is no infinite set of orthogonal idempotents. Then for any left annihilator L, there is an idempotent e such that $L = Re \oplus (L \cap R(1-e))$ and $L \cap R(1-e)$ is nil.*

Proof. If L is nil, there is nothing to prove. Suppose that L is not nil. Choose a non-nilpotent element x of L. By hypothesis there exists a positive integer n and $f \in E(R)$ such that $r(x^n) = fR$. Clearly $f \neq 1$ and $r(L) \subseteq r(x^n) = fR$, so that $L = l(r(L)) \supseteq l(fR) = R(1-f)$. Thus L contains a non-zero idempotent. Take a non-zero idempotent e in L such that $l(e)$ is minimal among the left annihilators of idempotents in L (cf. [8, Sublemma]). In a similar way as in the proof of [8, Theorem 1], we can prove that $L \cap R(1-e)$ is nil. Since $R = Re \oplus R(1-e)$ and $L \supseteq Re$, it is easy to see that $L = Re \oplus (L \cap R(1-e))$.

Corollary 1. *Let R be a generalized right p.p. ring in which there is no infinite set of orthogonal idempotents. If R has no non-zero nil right (or left) ideals, then R is a finite direct sum of prime Baer rings. In particular, R is a Bear ring.*

Proof. As we saw in the proof of Proposition 3, R has no non-zero nil right ideals if and only if it has no non-zero nil left ideals. By the above proposition R is a Baer ring. Hence by a result of Levy [1, Proposition 8.23, p. 123], R is a finite direct sum of prime rings.

The following lemma seems to be well known, but for convenience we give the proof. The proof is taken from that of [5, Theorem 2.3].

Lemma 2. *Let R be a prime ring of bounded index (of nilpotency). Then R has no infinite sets of orthogonal idempotents.*

Proof. Let R be of index n and suppose that R has non-zero orthogonal idempotents $e_1, e_2, \cdots, e_{n+1}$. Since R is a prime ring, we have $e_1Re_2R\cdots e_nRe_{n+1} \neq 0$. Hence there exist elements $a_1, a_2, \cdots, a_n$ of R such that $e_1a_1e_2a_2\cdots e_na_ne_{n+1} \neq 0$. Then, as is easily seen, the element $e_1a_1e_2+e_2a_2e_3+\cdots+e_na_ne_{n+1}$ is nilpotent of index $n+1$, a contradiction.

Corollary 2. *Let R be a prime ring of bounded index. If R is a generalized right p.p. ring, then R is a Baer ring.*

Proof. This follows from Lemma 1, Lemma 2 and Corollary 1.

Since a prime PI-ring is of bounded index, we have

Corollary 3. *Let R be a prime, generalized right p.p. ring with a polynomial identity. Then R is a Baer ring.*

Now we turn to characterizations of rings which have normal, classical right quotient rings. As in the commutative case, the following theorem is fundamental.

Theorem 1 (cf. [4, Theorem 1]). *Let R be a ring with normal, classical right quotient ring Q. Then the following are equivalent:*

1) *Every element of R is π-regular in Q.*

2) *For every zero-divisor x of R, there exists a positive integer n such that* $\mathrm{ann}_R(x^n)$ *coincides with* $\mathrm{ann}_R(x^{n+1})$ *and has a non-zero-divisor.*

3) *For every element x of R, there exists a positive integer n and a non-zero-divisor d of R such that $dx^n = x^nd = x^{2n}$.*

Proof. 1) $\Rightarrow$ 2). Assume 1) and let x be an arbitrary zero-divisor of R. By hypothesis there exists a positive integer n and an element $y \in Q$ such that $x^nyx^n = x^n$. Since Q is normal and yx^n is an idempotent, we have $yx^{2n} = x^n$, whence it follows that $Qx^n = Qx^{2n} = Qx^{n+1}$. Then we have

$r_R(x^n) = r_R(x^{n+1})$. Similarly we see that $l_R(x^n) = l_R(x^{n+1})$. Since yx^n and x^ny are central idempotents of Q, both $Qx^n = Qyx^n$ and $x^nQ = x^nyQ$ are two-sided ideals of Q containing x^n. Consequently $Qx^n = x^nQ$, so that $yx^n = x^ny$. Putting $e = 1 - yx^n$, we have $r_Q(x^n) = eQ = l_Q(x^n)$. Hence

$$r_R(x^n) = r_Q(x^n) \cap R = l_Q(x^n) \cap R = l_R(x^n).$$

This together with the above shows that $\mathrm{ann}_R(x^n) = \mathrm{ann}_R(x^{n+1})$. Writing $e = cd^{-1}$ ($c, d \in R$), we see that $c \in \mathrm{ann}_R(x^n)$. Let a be an element of $\mathrm{ann}_R(x^n)$. If $ac = 0$, then $a = ae = 0$. Now suppose that $ca = 0$. Noting that $ed = c$, we then have $da = eda = ca = 0$, whence $a = 0$. Thus we have proved that c is a non-zero-divisor of the ring $\mathrm{ann}_R(x^n)$.

2) $\Rightarrow$ 3). Assume 2) and let x be an arbitrary element of R. Since 3) is trivial in case x is a non-zero-divisor, we assume that x is a zero-divisor. Let z be a non-zero-divisor of $\mathrm{ann}_R(x^n)$. We contend that x^n+z is a non-zero-divisor of R. Let $a \in r_R(x^n+z)$. Then $x^{2n}a = x^n(x^n+z)a = 0$. Since $\mathrm{ann}_R(x^n) = \mathrm{ann}_R(x^{2n})$ by hypothesis, we have $a \in \mathrm{ann}_R(x^n)$. Accordingly $za = (x^n+z)a = 0$. Since z is a non-zero-divisor of $\mathrm{ann}_R(x^n)$, we get $a = 0$. Thus $r_R(x^n+z) = 0$. Similarly we have $l_R(x^n+z) = 0$. Hence x^n+z is a non-zero-divisor of R and we have $x^n(x^n+z) = x^{2n} = (x^n+z)x^n$.

3) $\Rightarrow$ 1). Assume 3) and let x be an arbitrary element of R. By hypothesis there exists a positive integer n and a non-zero-divisor $d \in R$ such that $dx^n = x^nd = x^{2n}$. Then we have $x^n = x^nd^{-1}x^n$. This completes the proof.

Lemma 3. *Let x be an element of a ring R and let n be a positive integer. If $r_R(x^n) = eR$ for some central idempotent e of R, then we have $r_R(x^n) = r_R(x^{n+1})$.*

Proof. Since the inclusion $r_R(x^n) \subseteq r_R(x^{n+1})$ is obvious, we show that $r_R(x^n) \supseteq r_R(x^{n+1})$. Let $a \in r_R(x^{n+1})$. Then $xa \in r_R(x^n) = eR$ and hence $xa = xae$. Therefore we have $x^na = x^{n-1}xa = x^{n-1}xae = x^nea = 0$, that is, $a \in r_R(x^n)$. Thus $r_R(x^{n+1}) \subseteq r_R(x^n)$, as desired.

Corollary 4. *Let R be a normal, generalized p.p. ring. Then for every element x of R, there exists a positive integer n and an idempotent e of R such that* $\mathrm{ann}_R(x^n) = eR$.

Proof. By hypothesis there exist positive integers l, m and $e, f \in E(R)$ such that $r_R(x^l) = eR$, $l_R(x^m) = fR$. Put $n = \max\{l, m\}$. By Lemma 3 we have $r_R(x^n) = eR$ and $l_R(x^n) = fR$. Since R is normal we readily obtain $e = f$ and $r_R(x^n) = l_R(x^n) = eR$.

Let R be a subring of a ring Q. If every element of Q has the form ab^{-1} with $a, b \in R$, we say that R is a *right order* in Q.

Lemma 4. *Let R be a right order in a normal ring Q. If R is a generalized right p.p. ring, then we have $E(Q) = E(R)$.*

Proof. Let $e \in E(Q)$ and write $e = ab^{-1}$ $(a, b \in R)$. By hypothesis there exists a positive integer n and $f \in E(R)$ such that $r_R(a^n) = fR$. Since $eb = a$ and e is central, we have $eb^n = a^n$, whence $e = a^n b^{-n}$. In view of [3, Lemma 1] we have

$$(1-e)Q = r_Q(e) = b^n r_R(a^n)Q = b^n fQ = fQ.$$

Hence $1-e = f$ and so $e = 1-f \in E(R)$, as desired.

We are now ready to prove our first main theorem which contains [4, Theorem 2].

Theorem 2. *Let R be a ring with normal, classical right quotient ring Q. Then the following are equivalent:*

1) *R is a generalized p.p. ring.*

2) *Every element of R is π-regular in Q and $E(Q) = E(R)$.*

Proof. 1) $\Rightarrow$ 2). This follows from Lemma 3, Corollary 4, Theorem 1 and Lemma 4.

2) $\Rightarrow$ 1). Assume 2) and let x be an arbitrary element of R. By hypothesis there exists a positive integer n and an element $y \in Q$ such that $x^n y x^n = x^n$. As was shown in the proof of Theorem 1, we have $Qx^n = x^n Q = eQ$, where $e = yx^n = x^n y \in E(Q) = E(R)$. Since $\mathrm{ann}_Q(x^n) = (1-e)Q$, we have

$$r_R(x^n) = r_Q(x^n) \cap R = (1-e)Q \cap R = (1-e)R.$$

Similarly we get $l_R(x^n) = (1-e)R$. Therefore R is a generalized p.p. ring.

Corollary 5 (cf. [4, Proposition 4]). *Let R be a ring with normal, classical right quotient ring Q. Suppose that R is a generalized p.p. ring. Then for any ideal $\mathfrak{A}$ of Q, $R/R \cap \mathfrak{A}$ is a generalized p.p. ring. In particular,* *if R is a commutative generalized p.p. ring*, then $R/P(R)$ *is a p.p. ring*.

Proof. For any element $a \in Q$ we write $\bar{a} = a+\mathfrak{A}$. The ring $\bar{Q} = Q/\mathfrak{A}$ contains a subring $\bar{R} = (R+\mathfrak{A})/\mathfrak{A}$ which is isomorphic to $R/R \cap \mathfrak{A}$. Let x be an arbitrary element of R. By Theorem 2 there exists a positive integer n and $e \in E(Q) = E(R)$ such that $Qx^n = x^nQ = eQ$. Then we have $\bar{Q}\bar{x}^n = \bar{x}^n\bar{Q} = \bar{e}\bar{Q}$ and $\bar{e} \in E(\bar{R})$. An easy verification shows that $r_{\bar{R}}(\bar{x}^n) = (\bar{1}-\bar{e})\bar{R} = l_{\bar{R}}(\bar{x}^n)$, whence $\bar{R}$ is a generalized p.p. ring. Since $P(R) = P(Q) \cap R$ in case R is commutative, the last assertion is immediate from Proposition 1.

A ring R is called *local* if $R/J(R)$ is a division ring.

Let R be a commutative ring and let $\mathfrak{q}$ be a proper ideal of R. Recall that $\mathfrak{q}$ is a *primary ideal* of R provided every zero-divisor of $R/\mathfrak{q}$ is nilpotent.

Although the following proposition is elementary, it enables us to extend several results in [4] and [6] to non-commutative rings.

Proposition 5. *Let R be a commutative ring and let Q be a classical quotient ring of R. Then the zero ideal (0) is a primary ideal of R if and only if Q is a local ring with nil Jacobson radical.*

Proof. Assume that (0) is a primary ideal of R and let $q = ab^{-1}$ (a, $b \in R$) be a non-unit of Q. Then a is a zero-divisor and $ac = 0$ for some non-zero $c \in R$. By hypothesis a is nilpotent and hence so is q. Consequently $1-q$ is a unit. It therefore follows that Q is a local ring with nil Jacobson radical.

Conversely assume that Q is a local ring with nil Jacobson radical. Let a, b be elements of R such that $ab = 0$ and suppose that b is not nilpotent. Then b is not in $J(Q)$ and so b is a unit in Q. Hence $a = 0$, thereby proving that (0) is a primary ideal of R.

Taking note of Theorem 2 and Proposition 5, we see that the next contains [4, Corollary 3].

Theorem 3. *Let R be a ring with normal, classical right quotient ring Q. Then the following are equivalent:*

1) *Q is a π-regular ring, $E(Q) = E(R)$ and R has no infinite sets of orthogonal idempotents.*

2) *R is a finite direct sum of rings whose classical right quotient rings are local rings with nil Jacobson radicals.*

Proof. 1) ⇒ 2). Assume 1). By hypothesis each non-nil right ideal of Q contains a non-zero idempotent and Q has no infinite sets of orthogonal idempotents. Then by [5, Theorem 2.1] the ring $\bar{Q} = Q/J(Q)$ is Artinian. On the other hand, each idempotent of $\bar{Q}$ can be lifted to an idempotent of Q, because $J(Q)$ is nil. Hence $\bar{Q}$ is normal. We can therefore write

$$\bar{Q} = \bar{Q}\bar{e}_1 \oplus \cdots \oplus \bar{Q}\bar{e}_n,$$

where $\bar{e}_1, \cdots, \bar{e}_n$ are orthogonal idempotents and each $\bar{Q}\bar{e}_i$ is a division ring. Let $e_1, \cdots, e_n$ be orthogonal idempotents of Q such that $e_i + J(Q) = \bar{e}_i$ ($i = 1, \ldots, n$). Since $e_1 + \cdots + e_n = 1$ and $E(Q) = E(R)$, we have

$$Q = Qe_1 \oplus \cdots \oplus Qe_n$$

and

$$R = Re_1 \oplus \cdots \oplus Re_n.$$

For each i, $\bar{Q}\bar{e}_i = Qe_i + J(Q)/J(Q)$ is isomorphic to $Qe_i/(Qe_i \cap J(Q)) = Qe_i/J(Qe_i)$, so that Qe_i is a local ring with nil Jacobson radical. It is easy to see that Qe_i is a classical right quotient ring of Re_i for every i.

2) ⇒ 1). Assume 2). We write

$$R = R_1 \oplus \cdots \oplus R_n,$$

where each R_i has a classical right quotient ring S_i which is a local ring with nil Jacobson radical. Since $S = S_1 \oplus \cdots \oplus S_n$ is also a classical right quotient ring of R, Q is isomorphic to S over R. By hypothesis each element of S_i is either nilpotent or invertible. Accordingly S is a π-regular ring and hence so is Q. Since every $E(S_i)$ consists of 0 and the identity element, we see that $E(S) = E(R)$ and hence that R has no infinite sets of orthogonal idempotents. We then have $E(Q) = E(R)$ and the proof is complete.

Remark (cf. [3, Theorem 6]). Let R be a right Ore ring (i.e. a ring which has a classical right quotient ring). Then the following are equivalent :

1) R is a normal p.p. ring which has no infinite sets of orthogonal idempotents.

2) R is a finite direct sum of right Ore domains.

Concerning characterizations of commutative generalized p.p. rings by means of localization, results of A.G. Naoum and Y. Hirano are summarized as follows (cf. [6, Theorem 1.9] and [4, Theorem 5]) :

Let R be a commutative ring and let Q be a classical quotient ring of R. Then the following are equivalent :

1) R is a generalized p.p. ring.

2) Q is a π-regular ring and for every prime ideal $\mathfrak{p}$ of R, (0) is a primary ideal of $R_{\mathfrak{p}}$.

3) Q is a π-regular ring and for every maximal ideal $\mathfrak{m}$ of R, (0) is a primary ideal of $R_{\mathfrak{m}}$.

To extend the above result to non-commutative rings, the notion of central localization is available. For the definition and properties of central localization, consult [7, §1.7]. Let R be an arbitrary ring with center C and let S be a multiplicative subset of C, that is, S is a subset of C which contains 1 and is closed under multiplication in R. We let $R_S = \{rs^{-1} \mid r \in R, s \in S\}$ be the localization of R by S. There is a canonical map of R into R_S given by $r \mapsto r1^{-1}$.

Proposition 6. *Let R be a ring with center C. Suppose that R has a normal, classical right quotient ring Q and that every element of R is π-regular in Q. Then for any multiplicative subset S of C, Q_S is a classical right quotient ring of R_S.*

Proof. First note that S is contained in the center of Q. Let ν: $Q \rightarrow Q_S$ be the canonical map. Let qs_0^{-1} ($q \in Q$, $s_0 \in S$) be an arbitrary element of Q_S. Writing $q = ab^{-1}$ with $a, b \in R$, we have $qs_0^{-1} = as_0^{-1}\nu(b)^{-1}$ with as_0^{-1}, $\nu(b) \in R_S$. Next let xs^{-1} ($x \in R$, $s \in S$) be an element of R_S. By hypothesis there exists a positive integer n and an element $y \in Q$ such that $x^n y x^n = x^n$. Then we have $(xs^{-1})^n \nu(ys^n)(xs^{-1})^n = (xs^{-1})^n$. As we saw in the proof of Theorem 1, $x^n y = yx^n$ and hence $e = (xs^{-1})^n \nu(ys^n) = \nu(x^n y) = \nu(yx^n) = \nu(ys^n)(xs^{-1})^n$, which is a central idempotent of Q_S. It follows that $(xs^{-1})^n Q_S = Q_S(xs^{-1})^n = eQ_S$. If $r_{R_S}(xs^{-1}) = 0$, then $r_{Q_S}(xs^{-1}) = 0$ and hence $e = 1$. Thus xs^{-1} is invertible in Q_S. This completes the proof.

Theorem 4. *Let R be a ring with center C and suppose that R has a normal, classical right quotient ring Q. Then the following are equivalent* :

1) *R is a generalized p.p. ring and for any maximal ideal $\mathfrak{m}$ of C, the set of nilpotent elements of Q_S is invariant under right multiplication by elements of Q_S, where S is the complement of $\mathfrak{m}$ in C.*

2) *Every element of R is π-regular in Q and for any maximal ideal $\mathfrak{m}$*

of C, Q_S is a local ring with nil Jacobson radical, where S is the complement of $\mathfrak{m}$ in C.

Proof. 1) $\Rightarrow$ 2). Assume 1). By Theorem 2 every element of R is π-regular in Q and $E(Q) = E(R)$ $(= E(C))$. Let $\mathfrak{m}$ be a maximal ideal of C and let S be the complement of $\mathfrak{m}$ in C. Let $\nu : Q \to Q_S$ be the canonical map. Set $K = \mathrm{Ker}\,\nu \cap R = \{a \in R \mid sa = 0 \quad \text{for some} \quad s \in S\}$. Let $e \in E(Q)$ $(= E(C))$. Then either $1-e \in S$ or $e \in S$. Hence it follows that either $e \in K$ or $1-e \in K$. Now let qs^{-1} $(q \in Q,\ s \in S)$ be an arbitrary element of Q_S and write $q = ab^{-1}(a, b \in R)$. By the proof of Theorem 1, there exists a positive integer n and $f \in E(Q)$ such that $a^nQ = Qa^n = fQ$. If $f \in K$, then $\nu(a)^n = 0$, whence $\nu(q) = \nu(a)\nu(b)^{-1}$ is nilpotent by hypothesis. Therefore qs^{-1} is nilpotent, so that $1-qs^{-1}$ is a unit. If $1-f \in K$, then $\nu(f) = 1$ and so $\nu(a)$ is a unit. Accordingly $\nu(q) = \nu(a)\nu(b)^{-1}$ is a unit and hence so is qs^{-1}. Thus Q_S is a local ring with nil Jacobson radical.

2) $\Rightarrow$ 1). By virtue of Theorem 2 it suffices to show that $E(Q) = E(R)$. Assume 2) and let $\mathfrak{m}$ be any maximal ideal of C. Let S be the complement of $\mathfrak{m}$ in C and let $\nu : Q \to Q_S$ be the canonical map. Let $e \in E(Q)$. Since $\nu(e)$ is an idempotent in the local ring Q_S, $\nu(e) = 0$ or 1. If $\nu(e) = 0$, then $se = 0$ for some $s \in S$. If $\nu(e) = 1$, then there exists $s' \in S$ such that $s'(1-e) = 0$, i.e. $s'e = s'$. Set $T = \{a \in C \mid ae \in R\}$. Obviously T is an ideal of C. By what we have shown above there are no maximal ideals of C containing T. Hence $T = C$ and so $e \in R$, as was to be shown.

Remark. The above proof shows that Theorem 4 remains true if we replace maximal ideals by prime ideals in 1) and 2).

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DEPARTMENT OF MATHEMATICS
SHINSHU UNIVERSITY

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