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Γ-RINGS AND MORITA EQUIVALENCE OF RINGS

Dedicated to Prof. Hirosi Nagao on his 60th birthday

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1. Introduction. Let S and T be (additive) subgroups of Hom(M,N) and of Hom(N,M) respectively for some additive groups M and N. For elements s in S and t in T, st indicates an element in Hom(N,N) such that $st(n) = s(t(n))$ for n in N. The set of all $\sum s_i t_i$ ($s_i \in S$ and $t_i \in T$) is denoted by ST. Similarly TS, STS and TST are defined. When $STS \subseteq S$ and $TST \subseteq T$, we say that (S,T) is a Γ-ring of homomorphisms (of M and N). Throughout this note, we assume that (S,T) is a Γ-ring of homomorphisms and denote it simply by G. For a Γ-ring G as above, TS is a subring of Hom(M,M), which we call the right operator ring of G. Similarly, ST is a subring of Hom(N,N), called the left operator ring of G. We denote them by R and L respectively. In [4], it was shown that when R and L contain the unities, they are Morita equivalent. Conversely, we can show that if two rings are Morita equivalent, they are the right and left operator rings of some Γ-ring. Thus, the correspondence of the right and left operator rings of a Γ-ring in a general case seems to be a generalization of Morita equivalence of rings. Let R and L be the right and left operator rings of a Γ-ring, which do not necessarily contain the unities. In § 2, the correspondence of R- and L-modules will be discussed. Let Q be a (non unital) ring and X a Q-module. If it satisfies (1) $QX = X$ and (2) $\{x \in X \mid Qx = 0\} = \{0\}$, then we say in this note that X is *properly generated* over Q. The main theorem obtained in § 2 is that if $R^2 = R$ and $L^2 = L$ then there is a one to one correspondence between properly generated R- and L-modules. This is a generalization of one of Morita duality theorems. In § 3, the correspondence between ideals of R and L will be discussed. Kyuno showed that there is a one to one correspondence between ideals of R and of L if R and L contain the unities ([2]) and that generally there is a one to one correspondence between prime ideals of R and of L ([3]). The first is naturally a consequence of one of Morita duality theorems (see [1]). We are going to generalize these results. Let I be an ideal of a ring Q. We define the upper and lower closures of I by $I^c = \{q \in Q \mid QqQ \subseteq I\}$ and $I_c = QIQ$ respectively. Note that I^c could be expressed as $Q^{-1}IQ^{-1}$. When $I = I^c$

(or $= I_c$), we say that I is closed above (or below). It will be shown that there is a one to one correspondence between ideals of R and of L, which are closed above (or below). If Q contains the unity, then every ideal of Q is closed above and below, which implies the first result of Kyuno. A prime ideal of a ring Q is easily seen to be closed above. We can show that an ideal which corresponds to a prime ideal in the above sense is also prime. This implies the second result of Kyuno. When a ring Q satisfies $Q^2 = Q$, we can derive some interesting facts. In this case, I^c is closed above and I_c is closed below. We will show that two ideals have the same upper closure if and only if they have the same lower closure. Thus, we can classify all ideals by their upper or lower closures. By the correspondence of upper closures (or lower closures), we have a one to one correspondence between classes of ideals of R and of L if $R^2 = R$ and $L^2 = L$. We can show that this correspondence does not depend on upper or lower closures.

2. Correspondence of modules. Let $G = (S, T)$ be a Γ-ring, and $R = TS$ and $L = ST$ the right and left operator rings of G. Let A be an R-module and U an L-module (both being considered as left modules.) A pair (A, U) is called a G-module if there exist a homomorphism σ of S to Hom-(A, U) (we denote s^σ by s') and a homomorphism τ of T to Hom(U, A) (t^τ is denoted by t') such that it satisfies $t's'(a) = (ts)a$, $s't'(u) = (st)u$, $s'[(t_1 s_1)a] = (st_1 s_1)a$ and $t'[(s_1 t_1)u] = (ts_1 t_1)u$, where s, $s_1 \in S$, t, $t_1 \in T$, $a \in A$ and $u \in U$. In the following, $s'(a)$ is denoted by sa, etc.

Proposition 1. *Suppose that A is an R-module such that $RA = A$. Then, there exists an L-module U such that (A, U) is a G-module.*

Proof. See [4, Proposition 1].

Next, we define a G-homomorphism of G-modules. Let (A, U) and (B, V) be G-modules. If f and g are an R-homomorphism of A to B and an L-homomorphism of U to V respectively such that $sf(a) = g(sa)$ and $tg(u) = f(tu)$, we say that a pair (f, g) is a G-homomorphism of (A, U) to (B, V).

Proposition 2. *Let (A, U) and (B, V) be G-modules. Suppose that $U = SA$ $(= \{\sum s_i a_i\})$ and that $\{v \in V \mid Lv = 0\} = \{0\}$. Then, for every R-homomorphism f of A to B, there exists a unique (up to isomorphism) L-homomorphism g of U to V such that (f, g) is a G-homomorphism of (A, U) to (B, V).*

Proof. Since $U = SA$, every element of U is expressed as $\sum s_i a_i$. We define g by $g(\sum s_i a_i) = \sum s_i f(a_i)$. To show that g is well defined, we must show that $\sum s_i a_i = 0$ implies $\sum s_i f(a_i) = 0$. Let $\sum s_i a_i = 0$. For an element st in L, we have $st \sum s_i f(a_i) = s \sum (ts_i) f(a_i) = sf(\sum (ts_i) a_i) = 0$. As L is generated by st, $L(\sum s_i f(a_i)) = 0$. Therefore, by the assumption of Proposition 2, $\sum s_i f(a_i) = 0$. It is easy to see that g is an L-homomorphism of U to V. We can also verify that $sf(a) = g(sa)$ and $tg(u) = f(tu)$. The uniqueness of g is also almost clear.

Proposition 3. *Let (A, U) be a G-module, and suppose that $U = SA$ and $\{u \in U \mid Lu = 0\} = \{0\}$. Then, U is uniquely determined (up to isomorphism) by A.*

Proof. Let (A, V) be another G-module satisfying the same conditions as (A, U). Let σ' be a homomorphism of S to $\mathrm{Hom}(A, V)$ in the definition of a G-module (A, V), but denote $s^{\sigma'}(a)$ by $s \circ a$ (in stead of sa which is used for (A, U)). Now we apply Proposition 2 for (A, U) and (A, V), where f is the identity mapping of A. The homomorphism g of U to V obtained in Proposition 2 is onto, because $g(\sum s_i a_i) = \sum s_i \circ f(a_i) = \sum s_i \circ a_i$. Replacing (A, U) and (A, V) by (A, V) and (A, U), we can conclude that g is an isomorphism.

Let X be a Q-module for a ring Q. Denote $N(X) = \{x \in X \mid Qx = 0\}$. If $Q^2 = Q$, then $N(X/N(X)) = 0$. For, let $\bar{x} \in N(X/N(X))$ where x is a representative of the coset $\bar{x}$, an element of $X/N(X)$. Then, $Q\bar{x} = \bar{0}$, so $Qx \subseteq N(X)$. Hence, $QQx = 0$. $Q^2 = Q$ implies $Qx = 0$, and hence $x \in N(X)$. Thus $\bar{x} = \bar{0}$.

Theorem 1. *Suppose that $R^2 = R$ and $L^2 = L$. Then there is a one to one correspondence between properly generated R-modules and properly generated L-modules. This correspondence $A \leftrightarrow U$ is given via a unique G-module (A, U) satisfying the conditions in Proposition 3. Moreover, if $A \leftrightarrow U$ and $B \leftrightarrow V$, then there exists an isomorphism of* $\mathrm{Hom}_R(A, B)$ *onto* $\mathrm{Hom}_L(U, V)$.

Proof. Let $\mathrm{PGM}(Q)$ denote the set of properly generated Q-modules. If $A \in \mathrm{PGM}(R)$, there exists an L-module W such that (A, W) is a G-module by Proposition 1. We may assume $SA = W$. Let $U = W/N(W)$. We show that (A, U) is a G-module. Here the homomorphism $S \to \mathrm{Hom}$-

(A, U) is defined naturally from the homomorphism $S \to \mathrm{Hom}(A, W)$. We define a homomorphism $\bar{\tau}$ of T to $\mathrm{Hom}(U, A)$ by $t^{\bar{\tau}}(u) = t'(w)$, where w is a representative of u and t' is an element of $\mathrm{Hom}(W, A)$ defined for the G-module (A, W). We need to show that $t^{\bar{\tau}}(u)$ dose not depend on the choice of representatives. It is enough to show that if $y \in N(W)$ then $t'(y) = 0$. So, let $y \in N(W)$. For an element $t_1 s$ ($t_1 \in T$ and $s \in S$) of R, $t_1 sty = 0$, since $sty \in Ly = 0$. Since $t_1 s$ generate R and $ty \in A \in \mathrm{PGM}(R)$, we have $ty = 0$, or $t'(y) = 0$ as requied. Now, it is not hard to verify that (A, U) is a G-module. $SA = U$ is clear. Also, by the remark before Theorem 1, $N(U) = 0$. We have also $LU = LSA = SRA = U$. Therefore, $U \in \mathrm{PGM}(L)$. The remaining parts of Theorem 1 follow from Propositions 2 and 3.

3. Correspondence of ideals of R and of L. For an ideal I of a ring Q, we define $I^c = \{q \in Q \mid QqQ \subseteq I\}$ and $I_c = QIQ$. I^c and I_c are ideals of Q and $I_c \subseteq I \subseteq I^c$. When $I^c = I$, we say that I is closed above. When $I_c = I$, we say that I is closed below. Note that if Q contains the unity, every ideal is closed above as well as below. Let A be an ideal of the right operator ring R of a Γ-ring. Define $A^* = \{l \in L \mid TlS \subseteq A\}$ and $A_* = SAT$. A^* and A_* are ideals of L. Similarly, for an ideal U of L, we can define U^* and U_*. It follows from the definitions that $(A^*)^* = A^c$ and $(A_*)_* = A_c$.

Proposition 4. *If A is closed above, then A^* is closed above. If A is closed below, then A_* is closed below.*

Proof. First, we show that $(A^*)^c = (A^c)^*$ and $(A_*)_c = (A_c)_*$. Let $u \in (A^*)^c$. Then $LuL \subseteq A^*$, or $TLuLS \subseteq A$. So, $RTuSR \subseteq A$, since $TL = RT$ and $LS = SR$. Therefore, $TuS \subseteq A^c$, and hence $(A^*)^c \subseteq (A^c)^*$. In a similar way, we can show $(A^c)^* \subseteq (A^*)^c$. Thus, $(A^*)^c = (A^c)^*$. On the other hand, $(A_*)_c = LSATL = SRART = (A_c)_*$. Now, suppose that A is closed above. Then, $(A^*)^c = (A^c)^* = A^*$, and A^* is closed above. Suppose that A is closed below. Then, $(A_*)_c = (A_c)_* = A_*$, and A_* is closed below.

Theorem 2. *There is a one to one correspondence between ideals of R and of L which are closed above (or below).*

Proof. For ideals A closed above, we consider the correspondence

$A \to A^* \to (A^*)^* = A^c = A$. For ideals A closed below, consider $A \to A_* \to (A_*)_* = A_c = A$.

Corollary (Kyuno). *There is a one to one correspondence between prime ideals of R and of L.*

Proof. A prime ideal is easily seen to be closed above. So, for a prime ideal A of R, we consider the correspondence $A \to A^*$. We must show that A^* is also prime. For it, let $UV \subseteq A^*$ with ideals U and V of L. Then, $TUSTVS \subseteq TUVS \subseteq TA^*S \subseteq A$. Therefore, $TUS \subseteq A$ or $TVS \subseteq A$, since A is prime. Then, $U \subseteq A^*$ or $V \subseteq A^*$, which proves that A^* is prime.

In the following, assume that Q satisfies $Q^2 = Q$. Then, for an ideal I of Q, $(I^c)^c = I^c$ and $(I_c)_c = I_c$, i.e., I^c (or I_c) is closed above (or below). Moreover, we have the following two identities : $(I_c)^c = I^c$ and $(I^c)_c = I_c$. For the former, Let $y \in I^c$. Then, $QyQ \subseteq I$, and so $QQyQQ \subseteq QIQ = I_c$. Since $QQ = Q$, this implies that $y \in (I_c)^c$, or $I^c \subseteq (I_c)^c$. The reverse inclusion is clear. So, $(I_c)^c = I^c$. For the latter, note $(I^c)_c = QI^cQ \subseteq I$, and hence $(I^c)_c \subseteq I_c$ as $(I^c)_c$ is closed below. Hence, $(I^c)_c = I_c$. We use these two identities to prove

Theorem 3. *Suppose $Q^2 = Q$. Let I and J be ideals of Q. Then the following conditions are equivalent.* (i) $I^c = J^c$, (ii) $I_c = J_c$, and (iii) $I_c \subseteq J \subseteq I^c$.

Proof. First, assume (i). Then, $I_c = (I^c)_c = (J^c)_c = J_c$, and (ii) holds. Similarly, we can show that (ii) implies (i). Next, suppose (i) and hence (ii) hold. Then, $I_c = J_c \subseteq J \subseteq J^c = I^c$, and (iii) holds. Lastly, assume (iii). Then, $(I_c)^c \subseteq J^c \subseteq (I^c)^c$, or $I^c \subseteq J^c \subseteq I^c$. Hence $I^c = J^c$, and (i) holds.

When one of the conditions in Theorem 3 holds for I and J, we say that I and J are c-equivalent. Then ideals of Q are classified by the c-equivalence. Denote the class of I by $|I|$. $|I|$ consists of all ideals that contain I_c and are contained in I^c.

Proposition 5. *Suppose that $L^2 = L$. Then, $|A^*| = |A_*|$ for every ideal A of R.*

Proof. It is enough to show that $(A^*)^c = (A_*)^c$. Let $u \in (A^*)^c$. Then, $TLuLS \subseteq A$, and so $STLuLST \subseteq SAT = A_*$, which implies $LuL = LLuLL \subseteq A_*$. Hence, $u \in (A_*)^c$. Thus, $(A^*)^c \subseteq (A_*)^c$. Clearly, $(A^*)^c \supseteq (A_*)^c$, and we have $(A^*)^c = (A_*)^c$.

Proposition 6. *Suppose that $L^2 = L$. If $|A| = |B|$ for ideals A and B of R, then $|A^*| = |B^*|$.*

Proof. We have $(A^*)^c = (A^c)^*$ and $(A_*)_c = (A_c)_*$ in the proof of Proposition 4. Therefore, $|(A^c)^*| = |A^*| = |A_*| = |(A_c)_*| = |(A_c)^*|$. Now, $|A| = |B|$ implies $A_c \subseteq B \subseteq A^c$, or $(A_c)^* \subseteq B^* \subseteq (A^c)^*$, from which we can conclude that $|A^*| = |B^*|$.

Theorem 4. *Suppose that $R^2 = R$ and $L^2 = L$. Then, there is a one to one correspondence between c-equivalent classes of ideals of R and of L. The correspondence is given by either $|A| \to |A^*|$ or by $|A| \to |A_*|$ of representatives A of classes, and it does not depend on the choice of representatives.*

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