Math. J. Okayama Univ. 26 (1984), 143—149

ON COMPLETELY TORSIONFREE MODULES

Dedicated to Professor Hirosi Nagao on his 60th birthday

HISAO KATAYAMA

In this paper we shall consider some module theoretic generalizations of completely torsionfree rings ([4, p.91]), and study those rings whose modules are completely torsionfree (CTF). We refer for the definitions and basic properties concerning preradicals and torsion theories to [8, Chap. VI].

All rings occurring are associative and possess an identity element. All modules are unitary left modules. Let R be a ring. We write ${}_RM$ to indicate that M is an object in the category R-mod of all left R-modules. We denote by $E(A)$ the injective hull of ${}_RA$. A module M is said to be strongly prime (SP) if, for each left exact preradical σ for R-mod, either $\sigma(M)=0$ or $\sigma(M)=M$. Also a module M is said to be 1SP if $\sigma(M)=0$ for every proper left exact preradical σ for R-mod. SP-modules was studied in Beachy and Blair [1] and 1SP-modules was studied in [4, p.114].

Now, we shall define completely torsionfree modules as follows:

Definition. A module M is called *completely torsionfree* (CTF) if, for each left exact radical ρ for R-mod, either $\rho(M)=0$ or $\rho(M)=M$. Further, M is called 1CTF (resp. 0CTF) if $\rho(M)=0$ (resp. $\rho(M)=M$) for every proper (resp. nonzero) left exact radical ρ for R-mod.

Remark 1. (1) Let ρ be a preradical for R-mod. Then $\rho(R)=R$ if and only if $\rho=1$, where 1 stands for the identity functor for R-mod. Hence a ring R is left CTF if and only if ${}_RR$ is 1CTF. Such a ring is also called left HRF in [2, p.160]. On the other hand, ${}_RR$ is 0CTF if and only if there exist only two left exact radicals for R-mod (i.e. R is left HRT ([2, p. 160])), or equivalently R is left ChC in the sense that, for all nonzero cyclic left R-modules C_1 and C_2, $\mathrm{Hom}_R(C_1, C_2)\neq 0$ holds ([4, p.96]). For characterizations of left CTF-rings, see [2, VI.1.E7] and [4, p.91], and for those of left ChC-rings, see [2, VI.2.1], [3], [4, p.96] and [7, Theorem 2.4].

(2) In [4, pp.117—119], we can find some connections between 1CTF-modules and CTF-rings.

(3) Hongan studied CTF-modules in [5, Theorem 1.3] and [6]. We notice that this result will also be obtained by using [2, I.6.4]. But we shall study in Theorem 5 the Hongan's result from our point of view. To do this and for further aims, we need next two lemmas.

Let ${}_RM$ be a fixed module. For each $x \in M$, we associate the left exact preradical σ_x and the left exact radical ρ_x for R-mod which are defined by

$$\sigma_x = \wedge \{\sigma \mid \sigma \text{ is a left exact preradical with } x \in \sigma(M)\}$$

and

$$\rho_x = \wedge \{\rho \mid \rho \text{ is a left exact radical with } x \in \rho(M)\}.$$

We also associate the set $\mathscr{L}_x$ of all left ideals I of R which contain $\mathrm{Ann}_R(Fx)$ for some finite subset F of R. Recall that there exists an order preserving correspondence between the left exact preradicals for R-mod and the left linear topologies on R, under which the left exact radicals correspond to the left Gabriel topologies.

Lemma 2 (cf. [4, pp. 89–90]). *Let x be an element of ${}_RM$. Then the set $\mathscr{L}_x$ is the smallest linear topology containing $\mathrm{Ann}_R(x)$. Moreover, the left exact preradical for R-mod corresponding to $\mathscr{L}_x$ is just σ_x.*

Proof. It is obvious that $\mathscr{L}_x$ is a filter. If $I \in \mathscr{L}_x$ and $s \in R$, then there exists a finite subset F of R with $\mathrm{Ann}_R(Fx) \subseteq I$, and so $\mathrm{Ann}_R(sFx) \subseteq (I: s)$. Hence $(I: s) \in \mathscr{L}_x$. Now, let $\mathscr{L}$ be an arbitrary left linear topology containing $\mathrm{Ann}_R(x)$. For each $I \in \mathscr{L}_x$, we have some finite subset F of R with $I \supseteq \mathrm{Ann}_R(Fx) = \bigcap_{r \in F} (\mathrm{Ann}_R(x) : r)$. By the assumption of $\mathscr{L}$, we have $I \in \mathscr{L}$.

Let $\mathscr{L}' \leftrightarrow \sigma_x$ and $\mathscr{L}_x \leftrightarrow \sigma'$ be the corresponding left linear topologies and left exact preradicals. Since $x \in \sigma_x(M)$, $\mathrm{Ann}_R(x) \in \mathscr{L}'$ and so $\mathscr{L}_x \subseteq \mathscr{L}'$. On the other hand, since $\mathrm{Ann}_R(x) \in \mathscr{L}_x$, we have $x \in \sigma'(M)$. Thus we obtain $\sigma_x \leqq \sigma'$ and hence $\mathscr{L}' \subseteq \mathscr{L}_x$. Therefore we must have $\mathscr{L}' = \mathscr{L}_x$ and $\sigma' = \sigma_x$.

Lemma 3. *Let x be an element of ${}_RM$. Then the smallest radical $\bar{\sigma}_x$ larger than σ_x is just ρ_x.*

Proof. Since $\bar{\sigma}_x$ is a left exact radical such that $x \in \sigma_x(M) \subseteq \bar{\sigma}_x(M)$, we have $\rho_x \leqq \bar{\sigma}_x$. Conversely, $x \in \rho_x(M)$ implies $\sigma_x \leqq \rho_x$ and so $\bar{\sigma}_x \leqq \rho_x$. Hence we have $\bar{\sigma}_x = \rho_x$.

Corollary 4. *Let x be an element of ${}_RM$, and let $\mathscr{G}_x$ denote the left Gabriel topology corresponding to ρ_x. Then a left ideal I of R belongs to $\mathscr{G}_x$ if and only if, for each proper left ideal J containing I, there exists $s \in R\backslash J$ such that $(J: s) \in \mathscr{L}_x$.*

Proof. Apply Lemma 3 with [8, Prop. VI.5.4].

For a module ${}_RE$, we define the radical k_E by $k_E(X) = \bigcap\{\mathrm{Ker}(\alpha) \mid \alpha \in \mathrm{Hom}_R(X, E)\}$ for each $X \in R$-mod. It is well known that k_E is left exact whenever E is injective and every left exact radical has the form k_E for some injective module ${}_RE$. Now, we shall prove the next

Theorem 5. *The following properties of a nonzero module ${}_RM$ are equivalent:*

(1) *M is CTF.*

(2) *If K is a proper submodule of M and x is a nonzero element of M, then the next condition holds:*

(*) *There exist $y \in M\backslash K$ and a finite subset F of R such that* $\mathrm{Ann}_R(Fx) \subseteq (K: y)$.

(3) *If K is a proper submodule of M and x is a nonzero element of M, then* $\mathrm{Hom}_R(Rx,\ E(M/K)) \neq 0$.

(4) *If K is a proper submodule of M, then $k_{E(M/K)}(M) = 0$, or equivalently M is cogenerated by $E(M/K)$.*

(5) ([5]) *If K is a non-trivial submodule of M, then there exists $v \in K$ with* $\mathrm{Hom}_R(Rv,\ M/K) \neq 0$.

Proof. (1) ⇒ (2). Note that the condition (*) is equivalent to the existence of $y \in M\backslash K$ with $(K: y) \in \mathscr{L}_x$, or equivalently $\sigma_x(M/K) \neq 0$ by Lemma 2. Assume that there exist a proper submodule K of M and a nonzero $x \in M$ such that $\sigma_x(M/K) = 0$. Since $\bar{\sigma}_x = \rho_x$ by Lemma 3, we have $\rho_x(M/K) = 0$. On the other hand, since $0 \neq x \in \rho_x(M)$, we must have $\rho_x(M) = M$ by (1), and so $\rho_x(M/K) = M/K$. This is a contradiction.

(2) ⇒ (3). Let K be a proper submodule of M, and take a nonzero $x \in M$. By the assumption, we have a finite subset $F = \{s_1, \cdots, s_n\}$ of R with $\mathrm{Ann}_R(Fx) \subseteq (K: y)$. Put $u = (s_1x, \cdots, s_nx) \in (Rx)^{(n)}$. Consider a cyclic submodule $U = Ru$ of $(Rx)^{(n)}$. Then we get a nonzero homomorphism $f: U \to M/K$ given by $f(au) = ay + K\ (a \in R)$. Extend f to $\bar{f} \in \mathrm{Hom}_R((Rx)^{(n)}, E(M/K))$. Hence we see $\mathrm{Hom}_R(Rx,\ E(M/K)) \neq 0$.

(3) ⇔ (4). Clear.

(3) $\Rightarrow$ (5). Let K be a non-trivial submodule of M. Take a nonzero $x \in K$. By the assumption, $\mathrm{Hom}_R(Rx, E(M/K)) \neq 0$ and so there exists $v \in Rx$ with $\mathrm{Hom}_R(Rv, M/K) \neq 0$.

(5) $\Rightarrow$ (1). Assume that $\rho(M)$ is a non-trivial submodule of M for some left exact radical ρ. Then there exists $v \in \rho(M)$ with $\mathrm{Hom}_R(Rv, M/\rho(M)) \neq 0$. This is a contradiction, because Rv is ρ-torsion and $M/\rho(M)$ is ρ-torsionfree.

Lemma 6. *The following conditions are equivalent for an element x of $_RM$:*

(1) $\rho_x = 1$.

(2) *For each proper left ideal J of R, there exist $s \in R\backslash J$ and a finite subset $F = \{s_1, \cdots, s_n\}$ of R with $\mathrm{Ann}_R(Fx) \subseteq (J : s)$. (We may take $n = 1$).*

(3) *If $_RQ$ is a nonzero (cyclic) module, then $\mathrm{Hom}_R(Rx, E(Q)) \neq 0$.*

Proof. (1) $\Leftrightarrow$ (2). Since $\rho_x = 1$ if and only if $\mathcal{G}_x \ni \{0\}$, this is clear from Corollary 4.

(2) $\Rightarrow$ (3). Consider a nonzero cyclic module $Q = R/J$, where J is a proper left ideal of R. By the assumption, there exist $s \in R\backslash J$ and a finite subset F of R with $\mathrm{Ann}_R(Fx) \subseteq (J : s)$. Now, the same argument as in the proof (2) $\Rightarrow$ (3) of Theorem 5 enables us to have $\mathrm{Hom}_R(Rx, E(Q)) \neq 0$.

(3) $\Rightarrow$ (2). For each proper left ideal J of R, we have $\mathrm{Hom}_R(Rx, E(R/J)) \neq 0$ by using (3). Hence there exists a cyclic submodule Ru of Rx with a nonzero $f \in \mathrm{Hom}_R(Ru, R/J)$. Put $u = s_1x$ and $f(u) = s+J$ ($s_1, s \in R$). Since f is well defined, we have the condition (2).

Theorem 7. *The following properties of a nonzero module $_RM$ are equivalent:*

(1) *M is 1CTF.*

(2) *$\rho_x = 1$ for every nonzero $x \in M$.*

(3) *If $_RE$ is a nonzero injective module, then $\mathrm{Hom}_R(Rx, E) \neq 0$ for every nonzero $x \in M$.*

Proof. (1) $\Rightarrow$ (2). Let x be a nonzero element of M. Since ρ_x is a left exact radical with $x \in \rho_x(M)$, we have $\rho_x(M) \neq 0$ and so $\rho_x = 1$ by (1).

(2) $\Rightarrow$ (3). Let x be a nonzero element of M and E a nonzero injective module. By using Lemma 6, we see $\mathrm{Hom}_R(Rx, E) \neq 0$.

(3) $\Rightarrow$ (1). Let ρ be a proper left exact radical for R-mod. Then

there exists an injective module $_RE$ such that $\rho = k_E$. Since $\rho \neq 1$, we see $E \neq 0$. Now, for any $x \in \rho(M)$, we see from $k_E(Rx) = Rx$ that $\mathrm{Hom}_R(Rx, E) = 0$. Hence, $\rho(M) = 0$ by (3) and therefore M is 1CTF.

Theorem 8. *The following properties of a nonzero module $_RM$ are equivalent:*

(1) *M is 0CTF.*

(2) *If $_RE$ is an injective module with* $\mathrm{Hom}_R(M, E) \neq 0$, *then E is a cogenerator.*

(3) *If N is a proper submodule of M, then $E(M/N)$ is a cogenerator.*

(4) *If N is a proper submodule of M and $_RS$ is a simple module, then* $\mathrm{Hom}_R(S, M/N) \neq 0$.

Proof. (1) $\Rightarrow$ (2). Let E be an injective module with $\mathrm{Hom}_R(M, E) \neq 0$. Then $k_E(M) \neq M$ and so, by (1), $k_E = 0$ where 0 stands for the zero functor for R-mod. Hence, for every $X \in R$-mod, $k_E(X) = 0$ induces that X is cogenerated by E.

(2) $\Rightarrow$ (3). Let N be a proper submodule of M. By (2), $\mathrm{Hom}_R(M, E(M/N)) \neq 0$ implies that $E(M/N)$ is a cogenerator.

(3) $\Rightarrow$ (1). Let ρ be a left exact radical such that $\rho(M) \neq M$. Then, for an injective module $_RE$ such that $\rho = k_E$, we have $\mathrm{Hom}_R(M, E) \neq 0$. Take any $f(\neq 0) \in \mathrm{Hom}_R(M, E)$. By $M/\mathrm{Ker}(f) \cong \mathrm{Im}(f) \subseteq E$, we see $E(M/\mathrm{Ker}(f)) \subseteq$ E. Therefore, by (3), E must be a cogenerator and so we have $\rho = 0$.

(3) $\Leftrightarrow$ (4). Recall that an injective module $_RE$ is a cogenerator if and only if $\mathrm{Hom}_R(S, E) \neq 0$ for all simple module $_RS$.

Corollary 9. *A simple module $_RS$ is 0CTF if and only if $E(S)$ is a cogenerator.*

Corollary 10. *The following statements are equivalent for a ring R:*

(1) *Every simple left R-module is 0CTF.*

(2) *All simple left R-modules are isomorphic.*

Proposition 11. *The class of 0CTF-modules forms a hereditary torsion class. On the other hand, the class of 1CTF-modules forms a torsionfree class closed under injective hulls.*

Proof. By the definition, a module is 0CTF if and only if it belongs to the intersection of all nonzero hereditary torsion classes. Clearly this intersection forms a hereditary torsion class. The remaining part is proved in the same way.

Clearly every simple left R-module is CTF. We remark that every projective (more generally, torsionless) left R-module is CTF (or 1CTF) if and only if R is left CTF. In the next theorem we shall characterize those rings whose (injective) left modules are CTF. As proved in [4, p.118], a ring with a simple 1CTF-module is just of this type.

Theorem 12. *The following assertions are equivalent for a ring R:*

(1) *Every left R-module is 1CTF.*

(2) *Every injective left R-module is 1CTF.*

(3) *Every simple left R-module is 1CTF.*

(4) *Every left R-module is 0CTF.*

(5) *Every injective left R-module is 0CTF.*

(6) *Every projective left R-module is 0CTF.*

(7) *Every left R-module is CTF.*

(8) *Every injective left R-module is CTF.*

(9) ([7]) *Every nonzero injective left R-module is a cogenerator.*

(10) *There exist only two left exact radicals for R-mod.*

(11) ([4]) *R is left ChC.*

(12) ([3]) *R is left semiartinian and all simple left R-modules are isomorphic.*

Proof. (1) ⇔ (2) ⇒ (3). Clear.

(3) ⇒ (9). Let ${}_RE$ be a nonzero injective module. For each simple module ${}_RS$, since k_E is a proper left exact radical, we have $k_E(S) = 0$. Thus $\mathrm{Hom}_R(S, E) \neq 0$ and hence E is a cogenerator.

(4) ⇔ (5) ⇒ (6). Clear.

(6) ⇒ (10). Since ${}_RR$ is 0CTF, this follows from Remark 1.

(7) ⇔ (8). Clear.

(7) ⇒ (12). Assume there exists a nonzero left R-module M with zero socle. We put the left exact preradical $\sigma = \mathrm{soc}$ and consider the left exact radical $\bar{\sigma}$. For a simple left R-module S, $\bar{\sigma}(M \oplus S) = S$ holds and so $M \oplus S$ is not CTF. Next, assume that S and T are non-isomorphic simple left R-modules. We put $\rho = k_{E(S)}$. Since $\rho(S) = 0$ and $\rho(T) = T$ by $\mathrm{Hom}_R(T, E(S)) = 0$, we have $\rho(S \oplus T) = T$, and hence $S \oplus T$ is not CTF.

(9) ⇔ (10). This was proved in [7, Theorem 2.4].

(10) ⇔ (11). This was proved in [4, p.96].
(10) ⇔ (12). This was proved in [3, Proposition 2].
(10) ⇒ (1),(4) and (7). Clear.

Finally we shall give an example which distinguishes 0CTF-modules, 1CTF-modules and CTF-modules.

Example 13. (1) Every 0CTF-module is CTF, but the converse is not true. In fact, let R be a left CTF but not ChC ring (for example, a two-sided simple ring with zero socle). Then ${}_RR$ is 1CTF but not 0CTF by Remark 1.

(2) Every 1CTF-module is CTF, but the converse is not true. To see this, let R be a local but not right perfect ring (for example, the ring of formal power series over a field). Then the (unique) simple module S is 0CTF by Corollary 10, but since R is not left ChC ([4, p.96]), S is not 1CTF by Theorem 12.

References

[1] J.A. Beachy and W.D. Blair: Rings whose faithful left ideals are cofaithful, Pacific J. Math. **58** (1975), 1–13.

[2] L. Bican, T. Kepka and P. Němec: Rings, Modules and Preradicals, Lecture Notes in Pure and Applied Math. **75**, Marcel Dekker, Inc., New York and Basel, 1982.

[3] B.J. Gardner: Rings whose modules form few torsion classes, Bull. Austral. Math. Soc. **4** (1971), 355–359.

[4] D. Handelman, K.R. Goodearl and J. Lawrence: Strongly prime and completely torsion-free rings, Carleton Math. Series **109**, Carleton University, 1974.

[5] M. Hongan: On strongly prime modules and related topics, Math. J. Okayama Univ. **24** (1982), 117–132.

[6] M. Hongan: Corrigendum to "On strongly prime modules and related topics" (This Journal, vol. 24, pp. 117–132), Math. J. Okayama Univ. **25** (1983), 195.

[7] H. Katayama: On rings for which various types of nonzero preradicals are cofaithful, Hokkaido Math. J. **12** (1983), 49–63.

[8] Bo Stenström: Rings of Quotients, Grundl. Math. Wiss. **217**, Springer-Verlag, Berlin-Heidelberg-New York, 1975.

Department of Mathematics
Yamaguchi University

(Received October 15, 1983)