Math. J. Okayama Univ. 26 (1984), 133–136

SOME COMMUTATIVITY RESULTS FOR RINGS WITH CERTAIN POLYNOMIAL IDENTITIES

AMIR H. YAMINI

Throughout the present paper, R will represent an associative ring (with or without 1), N the set of all nilpotent elements in R, and Z the center of R. Generalized commutators $[x, x_1, x_2, \cdots, x_k]$, for integers $k \geq 1$, are defined as follows : $[x, x_1] = xx_1 - x_1x$, if $k = 1$, and $[[x, x_1, x_2, \cdots, x_{k-1}], x_k]$, if $k \geq 2$. For $x_1 = x_2 = \cdots = x_k = y$, $[x, y, y, \cdots, y]$ is abbreviated as $[x, y]_k$. As is well known, if $[x, y]_2 = 0$ then $[x, y^m] = my^{m-1}[x, y]$ for any positive integer m. We denote by $Z(k)$ the set of all $x \in R$ such that $[x, x_1, x_2, \cdots, x_k] = 0$ for all $x_1, x_2, \cdots, x_k \in R$. Following [5], a ring R is said to be s-unital if for each x in R, $x \in xR \cap Rx$. As stated in [5], if R is an s-unital ring, then for any finite subset F of R there exists an element e in R such that $ex = xe = x$ for all $x \in F$. Such an element e will be called a pseudo-identity of F.

Now, let n be a fixed positive integer. Awtar [2] showed that if R is an $n!$-torsion free ring with 1 satisfying the polynomial identity $[x^n, y^n] = 0$ then it must be commutative. On the other hand, Bell [3] showed that an n-torsion free ring with 1 satisfying the same polynomial identity need not be commutative. More recently, Abu-Khuzam and Yaqub [1] proved that an n-torsion free ring with 1 satisfying the same polynomial identity must be commutative under some additional condition such as $x^ky^k - y^kx^k \in Z$ or $(xy)^k - (yx)^k \in Z$ with $(n, k) = 1$.

Let n be a fixed positive integer, and consider the following properties :

$(\mathrm{I})_n$ If $x, y \in R$ and $n[x, y] = 0$, then $[x, y] = 0$.

$(\mathrm{II})_n$ $[x^n, y^n] = 0$ for all $x, y \in R$.

The major purpose of this paper is to prove the following theorem which generalizes [1, Theorems 2 and 3] and [5, Theorem 1].

Theorem. *Let n, k be fixed positive integers with $(n, k) = 1$. Let R be an s-unital ring satisfying $(\mathrm{I})_n$ and $(\mathrm{II})_n$. Then the following are equivalent :*

(i) *R is commutative.*

(ii) *For every $x, y \in R$ there exists a positive integer $m = m(x, y)$ such that $[x^k, y^k]_m = 0$.*

(iii) *For every $x, y \in R$ there exists a positive integer $m = m(x, y)$*

such that $[(xy)^k-(yx)^k, y^k]_m = 0$.

(iv) *If* $x, y \in R$ *and* $x-y \in N$ *then either* $x^k-y^k \in Z(m)$ *with some positive integer* $m = m(x,y)$ *or both* x *and* y *commute with all elements in* N.

(v) *For every* $x \in R$ *and* $a \in N$ *there exists a positive integer* $m = m(x,a)$ *such that* $[\{x(1+a)\}^n-x^n(1+a)^n, x]_m = 0$ *(formally written).*

In preparation for proving our main theorem, we quote the following lemmas which are stated in [5].

Lemma 1. *Let* R *be an s-unital ring,* e *a pseudo-identity of* $\{a,b\} \subseteq R$. *If* $a^m b = (a+e)^m b$ *for some positive integer* m, *then* $b = 0$.

Lemma 2. *Let* R *be an s-unital ring satisfying* $(\mathrm{I})_n$ *and* $(\mathrm{II})_n$. *Then* $[a, x^n] = 0$ *for all* $a \in N$ *and* $x \in R$, N *is a commutative ideal containing the commutator ideal of* R, *and* $N^2 \subseteq Z$; *in particular,* $(x+ax)^m-(x+xa)^m = [a, x^m]$ *for all* $a \in N$, $x \in R$, *and positive integers* m.

Proof of Theorem. We start the proof by showing that either of (iii) and (iv) implies (ii). Suppose (iii). Let x, $y \in R$, and $a = [x^k, y^k] \in N$ (Lemma 2). Then $0 = [(y+ay)^k-(y+ya)^k, y^k]_m = [[a, y^k], y^k]_m = [x^k, y^k]_{m+2}$ for some positive integer m (Lemma 2), proving (ii). Next, suppose (iv). Let x, $y \in R$, and $a = [x^k, y^k]$, as above. Since $(a+y)-y \in N$, we see that $[a,y] = 0$ or $(a+y)^k-y^k \in Z(m)$ for some $m \geq 1$. If $(a+y)^k-y^k \in Z(m)$ then

$$[(a+y)^k-y^k, a+y] = -[y^k, a+y] = [a, y^k],$$

and therefore $[x^k, y^k]_{m+1} = [a, y^k]_m = 0$. Needless to say, if $[a,y] = 0$ then $[x^k, y^k]_2 = [a, y^k] = 0$, proving (ii).

Now we prove that (v) implies (ii). Let $a \in N$ and $y \in R$. Then there exists a positive integer m such that $[\{y(1+a)\}^n-y^n(1+a)^n, y]_m = 0$, that is, $[y^n(1+a)^n, y]_m = [\{y(1+a)\}^n, y]_m$. Since

$$[\{y(1+a)\}^n, y] = y[\{(1+a)y\}^n-\{y(1+a)\}^n] = y[a, y^n] = 0$$

by Lemma 2, we get $y^n[(e+a)^n, y]_m = [y^n(1+a)^n, y]_m = 0$, where e is a pseudo-identity of $\{a, y\}$. Similarly, $(y+e')^n[(e+a)^n, y]_{m'} = 0$ for some $m' \geq 1$, where e' is a pseudo-identity of $\{e, a, y\}$. Without loss of generality, we may assume that $m = m'$: $y^n[(e+a)^n, y]_m = 0$. But $N^2 \subseteq Z$ by Lemma 2. Hence $n[a,y]_m = 0$, and therefore $[a,y]_m = 0$. Now, let $x \in R$. Then $[x,y] \in N$ (Lemma 2), and we conclude that $[x,y]_{m+1} = 0$ for some

$m \geq 1$, proving (ii).

Finally, we prove that (ii) implies (i). In view of [4, Proposition 1], we may assume that R has 1. Suppose that $[r^k, s^k]_2 \neq 0$ for some $r, s \in R$. Then $[r^k, s^k]_m = 0$ and $[r^k, s^k]_{m-1} \neq 0$ for some $m > 2$. According to Lemma 2, $t = [r^k, s^k]_{m-2} \in N$ and $[t, x^n] = 0$ for all $x \in R$. Hence $[t, s^{kn}] = 0 = [t, (s^k+1)^n]$. Notice that $[t, s^k+1]_2 = [t, s^k]_2 = 0$. Then

$$ns^{k(n-1)}[t, s^k] = 0 = n(s^k+1)^{n-1}[t, s^k+1] = n(s^k+1)^{n-1}[t, s^k].$$

Thus by Lemma 1 and $(\mathrm{I})_n$, we obtain $[r^k, s^k]_{m-1} = [t, s^k] = 0$. This contradiction proves that $[x^k, y^k]_2 = 0$ for all $x, y \in R$. Now, let $u, v \in \{x^k \mid x \in R\}$. Since $[u^n, v]_2 = 0 = [u^n, v+1]_2$ and $[u^n, v^n] = 0 = [u^n, (v+1)^n]$, we obtain $nv^{n-1}[u^n, v] = 0 = n(v+1)^{n-1}[u^n, v]$, and therefore $[u^n, v] = 0$ by Lemma 1 and $(\mathrm{I})_n$. Noting that $[(u+1)^n, v]_2 = \sum_{i=0}^{n}\binom{n}{i}[u^{n-i}, v]_2 = 0$ by what proved just above, we can repeat the above argument for $u+1$ instead of u to see that $[(u+1)^n, v] = 0$. Combining these with $[v, u+1]_2 = 0 = [v, u]_2$, by repeated use of the above argument, we can see that $nu^{n-1}[u, v] = 0 = n(u+1)^{n-1}[u, v]$. Hence again by Lemma 1 and $(\mathrm{I})_n$, we get $[u, v] = 0$. This proves that R satisfies the polynomial identity $[x^k, y^k] = 0$. Hence, by [1, Theorem 2], R is commutative, proving (i).

Needless to say, every commutative ring R satisfies (ii)–(v). This completes the proof of the theorem.

Remark. The example of Johnsen, Outcalt and Yaqub cited in [1] shows that $(\mathrm{I})_n$ cannot be omitted in Theorem. Also, the existence of a finite non-commutative nil ring shows that the hypothesis that R is s-unital cannot be deleted. Finally, the following example shows that we cannot drop $(\mathrm{II})_n$:

Let R_m be the ring consisting of $m \times m$ matrices over Z of the form $\begin{pmatrix} a & & * \\ & \ddots & \\ 0 & & a \end{pmatrix}$

Here, $Z \neq Z(m-1) = R_m$ if $m > 2$.

Now, let R be the ring $R_1 \oplus R_2 \oplus R_3 \oplus \cdots$. Then R is an s-unital ring (without 1) and $Z(m) \subseteq Z(m+1)$ for all positive integers m. As is easily seen, R satisfies the condition (ii) in Theorem (for $k = 1$).

Acknowledgements. The author would like to express his appreciation and gratitude to his thesis advisor Professor Adil Yaqub for the help and for an excellent job of guiding him through the research leading to his thesis, as this paper is a part of it.

REFERENCES

[1] H. ABU-KHUZAM and A. YAQUB : n-torsion-free rings with commuting powers, Math. Japonica 25 (1980), 37-42.
[2] R. AWTAR : On the commutativity of non-associative rings, Publ. Math. Debrecen 22 (1975), 177-188.
[3] H.E. BELL : On rings with commuting powers, Math. Japonica 24 (1979), 473-478.
[4] Y. HIRANO, Y. KOBAYASHI and H. TOMINAGA : Some polynomial identities and commutativity of s-unital rings, Math. J. Okayama Univ. 24 (1982), 7-13.
[5] E. PSOMOPOULOS, H. TOMINAGA and A. YAQUB : Some commutativity theorems for n-torsion free rings, Math. J. Okayama Univ. 23 (1981), 37-39.

FACULTY OF SCIENCE AND COMPUTER
TEHRAN POLYTECHNIC
TEHRAN, IRAN

(Received October 11, 1983)
(Revised February 23, 1984)