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A COMMUTATIVITY THEOREM FOR ONE-SIDED S-UNITAL RINGS

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Throughout the present paper, R will represent a ring with center C, and N the set of all nilpotent elements of R. As for notations and terminologies used in the paper without mention, we follow [4].

In the proof of [4, Lemma 2 (3)], we need that if R is a left or right s-unital ring satisfying the condition (VII) in [4] then R is s-unital. But we did not handle the case that R is right s-unital. Because of this lack, the proof of [4, Theorem 2] is incomplete. The following theorem generalizes [2, Theorem] and [3, Theorem], and makes the proof of [4, Theorem 2] perfect, as well.

Theorem. *Let R be a right or left s-unital ring. Then the following are equivalent:*

1) *R is commutative.*

2) *For each pair of elements x, y of R there exist integers $m = m(x,y) > 1$ and $n = n(x,y) > 0$* such that $[x, x^n y - y^m x] = 0$, *and there exists a non-empty subset A of N such that for each $x \in R$ either $x \in C$ or there exists a polynomial $f(\lambda)$ in $\mathbf{Z}[\lambda]$ such that $x - x^2 f(x) \in A$.*

3) *For each $y \in R$ there exists an integer $m = m(y) > 1$ such that $[x, x^n y - y^m x] = 0 = [x, x^n y^m - y^{m^2} x]$ for all $x \in R$, where n is a fixed positive integer.*

4) *For each pair of elements x, y of R there exist relatively prime, positive integers $m = m(x,y)$ and $n = n(x,y)$ such that $(xy)^m = (yx)^m$ and $(xy)^n = (yx)^n$.*

5) *Given elements x_1, x_2, x_3, x_4, x_5 of R, there exists a positive integer $n = n(x_1, x_2, x_3, x_4, x_5)$ such that $[x_i^n, x_j^n] = 0 = [x_i^{n-1}, x_j^{n+1}]$ for all i, $j \in \{1, 2, 3, 4, 5\}$.*

In preparation for proving our theorem, we state first the following

Lemma. *Let R be a right (resp. left) s-unital ring. If for each pair of elements x, y of R there exists a positive integer $k = k(x,y)$ and an element $e' = e'(x,y)$ of R such that $e'x^k = x^k$ and $e'y^k = y^k$ (resp. $x^k e' = x^k$ and $y^k e' = y^k$), then R is an s-unital ring.*

Proof. Let x be an arbitrary element of R, and choose an element e of R such that $xe = x$. Then, by hypothesis, there exists a positive integer h and an element e' of R such that $e'(x+e)^h = (x+e)^h$ and $e'e^h = e^h$. Now, it is easy to see that

$$x = xe^{h-1} = e'(x+e)^h - (x+e)^h + xe^{h-1} \in Rx,$$

which proves that R is s-unital.

Corollary. *Let R be a right (resp. left) s-unital ring. If R satisfies any one of the following conditions, then R is an s-unital ring.*

1) *For each pair of elements x, y of R there exist positive integers $m = m(x,y)$ and $n = n(x,y)$ such that $[x, x^n y - y^m x] = 0$.*

2) *For each pair of elements x, y of R there exists a positive integer $n = n(x,y)$ such that $[x^n, y^n] = 0$.*

3) *For each pair of elements x, y of R there exists a positive integer $n = n(x,y)$ such that $(xy)^n = (yx)^n$.*

Proof. Let x, y be arbitrary elements of R. As is well known, we can find an element e of R such that $xe = x$ and $ye = y$ (resp. $ex = x$ and $ey = y$).

First, suppose that R satisfies 1). Then, there exist positive integers m, n; m', n' such that $e^m x^2 = [x, x^n e - e^m x] + x^2 = x^2$ and $e^{m'} y^2 = y^2$ (resp. $x^{n+1}e = [x, x^n e - e^m x] + x^{n+1} = x^{n+1}$ and $y^{n'+1}e = y^{n'+1}$), and therefore $e^{mm'}x^2 = x^2$ and $e^{mm'}y^2 = y^2$ (resp. $x^{n+n'}e = x^{n+n'}$ and $y^{n+n'}e = y^{n+n'}$). Hence, R is s-unital by Lemma.

Next, suppose that R satisfies 2). Let $n = n(x,e)$, and $n' = n(y,e)$. Then $e^n x^n = [x^n, e^n] + e^n x^n = x^n$ and $e^{n'} y^{n'} = y^{n'}$ (resp. $x^n e^n = [x^n, e^n] + x^n = x^n$ and $y^{n'} e^{n'} = y^{n'}$), and therefore $e^{nn'} x^{n+n'} = x^{n+n'}$ and $e^{nn'} y^{n+n'} = y^{n+n'}$ (resp. $x^{n+n'} e^{nn'} = x^{n+n'}$ and $y^{n+n'} e^{nn'} = y^{n+n'}$). Hence, R is s-unital by Lemma.

Finally, suppose that R satisfies 3). Let $n = n(x,e)$, and $n' = n(y,e)$. Then $ex^n = (ex)^n = (xe)^n = x^n$ and $ey^{n'} = y^{n'}$ (resp. $x^n e = x^n$ and $y^{n'} e = y^{n'}$), and therefore $ex^{n+n'} = x^{n+n'}$ and $ey^{n+n'} = y^{n+n'}$ (resp. $x^{n+n'}e = x^{n+n'}$ and $y^{n+n'}e = y^{n+n'}$). Hence, again by Lemma, R is s-unital.

We are now ready to complete the proof of our theorem.

Proof of Theorem. Every commutative ring satisfies 2)—5) trivially.

2) $\Rightarrow$ 1) and 3) $\Rightarrow$ 1). First, in case R has 1, the proof of both

implications given in the proof of [4, Theorem 2] is still valid. Since R is s-unital by Corollary, R is commutative by the proof of [1, Proposition 1].

4) $\Rightarrow$ 1) and 5) $\Rightarrow$ 1). In case R has 1, [2, Theorem] and [3, Theorem] show that R is commutative. Again by Corollary, R is s-unital. Hence R is commutative by [1, Proposition 1].

Remark 1. Let n be a positive integer, and consider the following ring-properties (see [1]):

$P_1(n)$ $(xy)^n = x^n y^n$ and $(xy)^{n+1} = x^{n+1}y^{n+1}$ for all $x, y \in R$.
$P_3(n)$ $(xy)^n = (yx)^n$ for all $x, y \in R$.
$P_4(n)$ $[x,(xy)^n] = 0$ for all $x, y \in R$.
$P_5(n)$ $[x,(yx)^n] = 0$ for all $x, y \in R$.
$P_7(n)$ $[x,y^n] = [x^n,y]$ for all $x, y \in R$.
$P_8(n)$ There is a polynomial $f(\lambda) \in \mathbf{Z}[\lambda]$ such that $[x,y^n] = [f(x),y]$ for all $x, y \in R$.
$P_9(n)$ $[x,(x+y)^n - y^n] = 0$ for all $x, y \in R$.
$P_{10}(n)$ $[x^n,y^n] = 0$ for all $x, y \in R$.

In view of [1, Proposition 3 (ii)], we see that if $n > 1$ then $P_7(n) \Leftrightarrow P_8(n) \Leftrightarrow P_9(n) \Rightarrow P_{10}(n^{\alpha})$ for some positive integer α. Furthermore, as was shown in [1, Remark 4], if an s-unital ring R has the property $P_i(m) \wedge P_j(n)$ for some positive integers $i, j \in I = \{1, 3, 4, 5, 7, 8, 9, 10\}$, $m > 1$ and $n > 1$ with $(m, n) = 1$, then R is commutative. Now, these results together with Corollary enable us to see the following: *If a right or left s-unital ring R has the property $P_i(m) \wedge P_j(n)$ for some positive integers $i \in I$, $j \in \{3, 7, 8, 9, 10\}$, $m > 1$ and $n > 1$ with $(m, n) = 1$, then R is commutative.*

Remark 2. If R is a left (resp. right) s-unital ring having the property $P_4(n)$ (resp. $P_5(n)$) then Lemma shows that R is s-unital. However, if $R = \begin{pmatrix} K & 0 \\ K & 0 \end{pmatrix}$ (K a field), then R has a right identity element and has the property $P_1(n) \wedge P_4(n)$ for any positive integer n, but R is not left s-unital. If, furthermore, $K = \mathrm{GF}(2)$, then $x \cdot x^2 = x^2$ for all $x \in R$ (cf. Lemma).

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