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TWO COMMUTATIVITY PROPERTIES FOR RINGS

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Throughout the present paper, R will represent an associative ring with center C, and N the set of all nilpotent elements in R. A ring R is called left (resp. right) s-unital if $x \in Rx$ (resp. $x \in xR$) for every $x \in R$; R is called s-unital if R is both left and right s-unital.

Let n, m be fixed positive integers, and A a non-empty subset of R. We consider the following conditions:

(I-A) For each $x \in R$ there exists a polynomial $f(t)$ in $\mathbf{Z}[t]$ such that $x-x^2f(x) \in A$.

(I_n-A) For each $x \in R$ there exists a polynomial $f(t)$ in $\mathbf{Z}[t]$ such that $nx-x^2f(x) \in A$.

(I'_n-A) For each $x \in R$, either $x \in C$ or there exists a polynomial $f(t)$ in $\mathbf{Z}[t]$ such that $nx-x^2f(x) \in A$.

(II-A) If x, $y \in R$ and $x-y \in A$, then either $x^2 = y^2$ or x and y both belong to the centralizer $V_R(A)$ of A in R.

$Q(n)$ For any x, $y \in R$, $n[x,y] = 0$ implies $[x,y] = 0$.

$P'_6(m)$ For each pair of elements x, y in R there exists a positive integer $i = i(x,y)$ such that $[x, y^{m^i}] = 0$.

(C) For each pair of elements x, y in R there exist $f(t)$, $g(t) \in \mathbf{Z}[t]$ such that $[x-x^2f(x), y-y^2g(y)] = 0$.

Recently, by making use of a theorem of Bell [1, Theorem 2], Cherubini and Varisco [2, Theorem 1] proved that if R contains a commutative subset A for which R satisfies (I-A) and (II-A) then R is commutative. Soon after, in the preceding paper, the second author proved directly the same ([7, Theorem 1]).

The present objective is to prove the following commutativity theorem, which generalizes [7, Theorem 1], and leads to a generalization of [3, Corollary 1] and [7, Corollary 1].

Theorem 1. *The following are equivalent:*

1) *R is commutative.*

2) *R satisfies $Q(n)$ and contains a commutative subset A for which R satisfies* (I_n-A) *and* (II-A).

3) *R satisfies $Q(n)$ and N contains a commutative subset A for which*

R satisfies (I'_n-A) *and* (II-A).

In preparation for proving our theorem, we state the following four lemmas.

Lemma 1. (1) *If R satisfies* (II-A), *then* $[a,x^2]=0$ *for every* $a \in A$ *and* $x \in R$.

(2) *If R has* 1 *and satisfies* (II-A), *then* $A \subseteq C \cup \{x \in R \mid 2x=0\}$.

(3) *Suppose R satisfies* (II-A). *If either* $N \subseteq R^2$ *or R is semiprime, then A is commutative.*

(4) *If R satisfies* (I'_n-A) *then for each* $u \in N$ *there exists a positive integer k such that* $n^k u \in (A \cup C)^{-}$, *the additive subsemigroup of R generated by* $A \cup C$.

(5) *If R satisfies* (I'_n-A), (II-A) *and* $Q(n)$ *then R is normal, that is, every idempotent of R is central.*

(6) *Suppose a left* (*or right*) *s-unital ring R satisfies* $P'_6(m)$. *If* $(n,m)=1$ *then R satisfies* $Q(n)$.

Proof. (1) and (2) are given in [7, Lemma 1], and (4) can be seen by a trivial induction on nilpotency index.

(3) Suppose, to the contrary, that there exists $b \in A \setminus V_R(A)$. Let a be an arbitrary element of A. Since for every $x \in R$, either x or $x+b$ is in $R \setminus V_R(A)$, we have $a^2=(a+b)^2=b^2=0$. Thus, $A \subseteq N$ and $bx+xb=(x+b)^2-x^2=0$. Since $bxy=-xby=xyb$ for any x, $y \in R$, if $N \subseteq R^2$ then we have a contradiction $b \in V_R(N) \subseteq V_R(A)$. Since $bxb=-b^2x=0$ for every $x \in R$, if R is semiprime then we have a contradiction $b=0$.

(5) Let e be an idempotent of R. Let x be an arbitrary element of R, and set $u=xe-exe$. Obviously, $u^2=0$, and hence $nu \in A \cup C$ by (I'_n-A). Since $[nu,e]=0$ by Lemma 1 (1), we get $u=[u,e]=0$, i.e., $xe=exe$. Similarly, we can show that $ex=exe$, and therefore $ex=xe$.

(6) Let $I=\{x \in R \mid m^j x=0$ for some positive integer $j\}$. Then I is an ideal of R and R/I is m-torsion free. Since R/I satisfies $P'_6(m)$ (and is s-unital), R/I is commutative by [6, Theorem 2 (1)]. Hence, for each pair of elements x, y in R, there exists a positive integer $k=k(x,y)$ such that $m^k[x,y]=0$. If $n[x,y]=0$ then $(n,m)=1$ implies $[x,y]=0$, which proves that R satisfies $Q(n)$.

Lemma 2. *Let A be a commutative subset of R.*

(1) *If* R *satisfies* $(\mathrm{I}'_n\text{-}A)$ *and* $Q(n)$ *then* $N \subseteq V_R(A)$.

(2) *If* R *satisfies* $(\mathrm{I}'_n\text{-}A)$ *and* $Q(n)$ *then* R *does* (C). *If, furthermore,* $A \subseteq C$ *then* R *is commutative.*

Proof. (1) Let $u \in N$, and $n^k u \in (A \cup C)^+$ (Lemma 1 (4)). Since A is commutative, we have $n^k[u,a] = [n^k u, a] = 0$ for all $a \in A$. Hence $[u,a] = 0$ by $Q(n)$, which proves that $N \subseteq V_R(A)$.

(2) Let $x, y \in R$. Then, by $(\mathrm{I}'_n\text{-}A)$, there exist $h(t), k(t) \in \mathbf{Z}[t]$ such that $n^2x - n^2x^2h(nx)$, $n^2y - n^2y^2k(ny) \in A \cup C$. Since A is commutative, we have $n^4[x - x^2h(nx), y - y^2k(ny)] = 0$, and hence $Q(n)$ gives (C). If, furthermore, $A \subseteq C$ then $n^2[x - x^2h(nx), y] = 0$, so that $x - x^2h(nx) \in C$ by $Q(n)$. Hence R is commutative by a theorem of Herstein [4, Theorem 19].

Lemma 3. *Let* R *be a normal, subdirectly irreducible ring. Suppose* N *contains a commutative subset* A *for which* R *satisfies* $(\mathrm{I}'_n\text{-}A)$. *If* R *satisfies* $Q(n)$ *and* $A \not\subseteq C$, *then* R *has* 1 *and the characteristic of* R *is a power of a prime.*

Proof. Choose $a \in A$ and $x \in R$ such that $[a,x] \neq 0$. By $(\mathrm{I}'_n\text{-}A)$, $n^3x - (n^2x)^2f(n^2x) \in A \subseteq N$ for some $f(t) \in \mathbf{Z}[t]$. Then, we can easily see that $(n^3x)^i = (n^3x)^{2i}g(x)$ with some $i > 0$ and $g(t) \in \mathbf{Z}[t]$. Since $n^3x \notin N$ by Lemma 2 (1), $e = (n^3x)^i g(x)$ is a non-zero central idempotent, and therefore $e = 1$ and n^3x is invertible. Then, noting that $1 = n^{3i}x^i g(x)$, we see that x^{-1} is integral over $\mathbf{Z}\cdot 1$. Since a cannot commute with both $2x^{-1}$ and $3x^{-1}$, there exists an integer $k > 1$ such that $[a, kx^{-1}] \neq 0$. Then, by the above argument, $(kx^{-1})^{-1}$ is integral over $\mathbf{Z}\cdot 1$, and hence $(k\cdot 1)^{-1} = (kx^{-1})^{-1}x^{-1}$ also is integral over $\mathbf{Z}\cdot 1$. Obviously, this implies that the additive order of 1 is non-zero, and therefore a power of a prime.

Lemma 4. *Let* R *be a subdirectly irreducible ring with* 1. *Suppose* R *(resp.* N*) contains a subset* A *for which* R *satisfies* $(\mathrm{I}_n\text{-}A)$ *(resp.* $(\mathrm{I}'_n\text{-}A)$*) and* $(\mathrm{II}\text{-}A)$. *If* R *satisfies* $Q(n)$ *then* R *is commutative.*

Proof. In view of Lemma 1 (3), A is commutative. By Lemma 2 (2), it suffices to show that $A \subseteq C$. Suppose, to the contrary, that $[a,x] \neq 0$ for some $a \in A$ and $x \in R$. By hypothesis, $(nx)^2 = (x^2f(x))^2$ with some $f(t) \in \mathbf{Z}[t]$. Since R satisfies (C) (Lemma 2 (2)), N forms a (commutative) nil ideal containing the commutator ideal D of R, by a theorem of

Chacron (see, e.g., [5, Theorem 1]). By (I_n-A), $2n^2-4n^2g(2n) \in A$ with some $g(t) \in Z[t]$, and so we can find a non-zero integer k such that $k\cdot 1 \in A$. Noting that $(k\cdot 1+x)^2 = x^2$ and $(k\cdot 1-x)^2 = (-x)^2$ by (II-A), we obtain $2k^2\cdot 1 = 0$. This implies that the subdirectly irreducible ring R is of characteristic p^α with some prime p. (In case R satisfies (I'_n-A) and (II-A) for some $A \subseteq N$, Lemma 3 together with Lemma 1 (5) shows that R is of characteristic p^α.) Combining this with $2a = 0$ (Lemma 1 (2)), we see that $p = 2$. Obviously, $(n,2) = 1$ by $Q(n)$, and the non-zero subring of R/N generated by $x+N$ is a finite field GF(2^β) (Lemma 2 (1) and Lemma 1 (5)), and therefore $x^{2^\beta}-x \in N \subseteq V_R(A)$ (Lemma 2 (1)). But, $[x^2,a] = 0$ by Lemma 1 (1), and we have a contradiction $[x,a] = [x^{2^\beta},a]-[x^{2^\beta}-x,a] = 0$.

We are now ready to complete the proof of our theorem.

Proof of Theorem 1. It is enough to show that each of 2) and 3) implies 1). To see this, by Lemma 2 (2), it suffices to show that $A \subseteq C$. Suppose, to the contrary, that $[a,x] \neq 0$ for some $a \in A$ and $x \in R$. It is clear that $[a,n^3x] \neq 0$ and $n^3x \notin N$ (Lemma 2 (1)). By (I'_n-A), there exists $f(t) \in Z[t]$ such that $n^3x-n^4x^2f(n^2x) \in A$. Since $[a,n^3x] \neq 0$, we have $(n^3x)^2 = \{n^4x^2f(n^2x)\}^2 = (n^3x)^2\{nxf(n^2x)\}^2 = (n^3x)^2\{nxf(n^2x)\}^6 = (n^3x)^4g(x)$ with some $g(t) \in Z[t]$. Then $e = (n^3x)^2g(x)$ is a non-zero central idempotent (Lemma 1 (5)). As is well known, R is a subdirect sum of subdirectly irreducible rings R_γ with epimorphisms $\psi_\gamma: R \to R_\gamma$ ($\gamma \in \Gamma$). Choose $\delta \in \Gamma$ such that $[\psi_\delta(a), \psi_\delta(n^3x)] \neq 0$. Then $\psi_\delta(n^6x^2) \neq 0$. For, otherwise, $n^3x-n^4x^2f(n^2x) \in A$ and $n^4x^2f(n^2x)-n^6x^2\{xf(n^2x)\}^2h(n^3x^2f(n^2x)) \in A \cup C$ with some $h(t) \in Z[t]$ give a contradiction $\psi_\delta(n^3x) \in \psi_\delta((A \cup C)^+)$. Hence, the non-zero central idempotent $\psi_\delta(e)$ is the identity element 1 of R_δ and $n\cdot 1$ is invertible. In particular, R_δ satisfies $Q(n)$. Hence, R_δ is commutative by Lemma 4. This contradiction shows that $A \subseteq C$.

Combining Theorem 1 with Lemma 1 (3), we readily obtain the following generalization of [3, Corollary 1] and [7, Corollary 1].

Corollary 1. *Let R be a ring with $Q(n)$. Suppose R (resp. N) contains a subset A for which R satisfies* (I_n-A) *(resp.* (I'_n-A)) *and* (II-A). *If either $N \subseteq R^2$ or R is semiprime, then R is commutative.*

Corollary 2. *Let $(n,m) = 1$. If R is left (or right) s-unital, then the following are equivalent:*

1) *R is commutative.*

2) *R satisfies $P'_6(m)$ and contains a subset A for which R satisfies $(\mathrm{I}_n\text{-}A)$ and $(\mathrm{II}\text{-}A)$.*

3) *R satisfies $P'_6(m)$ and N contains a subset A for which R satisfies $(\mathrm{I}'_n\text{-}A)$ and $(\mathrm{II}\text{-}A)$.*

Proof. It suffices to show that each of 2) and 3) implies 1). By Lemma 1 (3) and (6), A is commutative and R satisfies $Q(n)$. Hence, R is commutative by Theorem 1.

We conclude this paper with the following example which shows that Theorem 1 is not necessarily true if we change $(\mathrm{II}\text{-}A)$ to $(\mathrm{II}\text{-}A)_3$: if x, $y \in R$ and $x-y \in A$ then either $x^3 = y^3$ or x and y both belong to $V_R(A)$.

Example 1. Consider the subsets

$$K = \left\{0, \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \begin{pmatrix}0 & 1\\ 1 & 1\end{pmatrix}, \begin{pmatrix}1 & 1\\ 1 & 0\end{pmatrix}\right\} \text{ and } H = \left\{0, \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}, \begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}\right\}$$

of $M_2(\mathrm{GF}(2))$. Then $R = \left\{\begin{pmatrix}\chi & \eta\\ 0 & \chi\end{pmatrix} \mid \chi \in K,\ \eta \in H\right\}$ is a non-commutative algebra over GF(2), and $N = \begin{pmatrix}0 & 0\\ H & 0\end{pmatrix}$ squares to zero. It is easy to see that $3x - x^4 = x + x^4 \in N$ for all $x \in R$, and hence $(\mathrm{I}_3\text{-}N)$. Further, it can be verified that R satisfies $(\mathrm{II}\text{-}N)_3$.

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