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A COMMUTATIVITY THEOREM FOR RINGS

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Throughout the present paper, R will represent a ring with center C, N the set of all nilpotent elements of R, and D the commutator ideal of R. A ring R is called left (resp. right) s-unital if $x \in Rx$ (resp. $x \in xR$) for every $x \in R$; R is called s-unital if R is both left and right s-unital. As stated in [3] and [5], if R is s-unital (resp. left or right s-unital), for any finite subset F of R there exists an element e in R such that $ex = xe = x$ (resp. $ex = x$ or $xe = x$) for all $x \in F$. We shall use freely the following well known result: Let $r, s \in R$, and k a positive integer. If there exists an element e in R such that $er = re = r$, $es = se = s$ and $r^k s = 0 = (r+e)^k s$ then $s = 0$.

Our objective is to prove the following

Theorem. *Let m, n be fixed non-negative integers. Suppose that R satisfies the polynomial identity*

$$(1) \qquad x^n[x,y]-[x,y^m] = 0.$$

(a) *If R is left s-unital, then R is commutative except the case $m = 1$ and $n = 0$.*

(b) *If R is right s-unital, then R is commutative except the cases $m = 1$ and $n = 0$; $m = 0$ and $n > 0$.*

In preparation for proving our theorem, we state three lemmas.

Lemma 1. *Let P be a ring-property which is inherited by every subring and every homomorphic image, and let $f(x_1, \cdots, x_k)$ be an element of the free ring $\mathbf{Z}\{x_1, \cdots, x_k\}$ generated by $x_1, \cdots, x_k$. If every ring with 1 having the property P satisfies the polynomial identity $f(x_1, \cdots, x_k) = 0$, then every left (resp. right) s-unital ring having the property P satisfies the polynomial identity $f(x_1, \cdots, x_k)x_{k+1} = 0$ (resp. $x_{k+1}f(x_1, \cdots, x_k) = 0$).*

Proof. Let R be a left s-unital ring having the property P. Let r be an arbitrary element of R, and set $S(r) = \{s \in R \mid l_R(r) \subseteq l_R(sr)\}$. Obviously, $S(r)$ is a subring of R, and $l_{S(r)}(r)$ is a (two-sided) ideal of $S(r)$. Choose e such that $er = r$. Then, e is in $S(r)$ and $\bar{e} = e + l_{S(r)}(r)$ is a right identity element of $\overline{S(r)} = S(r)/l_{S(r)}(r)$. In fact, e is the identity element of $\overline{S(r)}$. To see this, let s be an arbitrary element of $S(r)$, and choose e' such that $e's = s$ and $e'e = e$ (and hence $e'r = r$). Since $e\text{-}e' \in l_R(r)$, we get $(es-s)r = (e-e')sr = 0$, namely $\overline{es} = \overline{s}$. Hence, by hypothesis, $\overline{S(r)}$ satisfies the polynomial identity $f(x_1,\cdots,x_k) = 0$. Now, let $r_1,\cdots,r_{k+1}$ be arbitrary elements in R, and choose e^* such that $e^*r_i = r_i$ $(i = 1,\cdots, k+1)$. Obviously, every r_i is in $S(e^*)$, and therefore, as was claimed just above, $f(r_1,\cdots,r_k)r_{k+1} = f(r_1,\cdots,r_k)e^*r_{k+1} = 0$ (in $S(e^*)$).

Lemma 2. *Let R be a ring with* 1, *and let m, n be fixed non-negative integers. Suppose that R satisfies* (1). *If $(m, n) \neq (1, 0)$, then R is commutative.*

Proof. If $m = 0$, the assertion is immediate. If $m = 1$, then (1) becomes

$$[x,y]+x^nyx-x^{n+1}y = 0. \tag{2}$$

Hence, by [2, Theorem], R is commutative provided $n > 0$. (In case $n = 0$, (2) is superfluous.) We assume henceforth that $m > 1$. Let $a \in N$ with $a^l = 0$, and choose t such that $m^t > l$. Then, an easy induction shows that $x^{tn}[x,a] = [x,a^{m^t}] = 0$ for all $x \in R$. Hence, $[x,a] = 0$, namely $N \subseteq C$. Now, observe that D is a nil ideal of R by [1, Proposition 2], since $x = e_{11}$ and $y = e_{12}$ fail to satisfy (1). We obtain therefore $D \subseteq N \subseteq C$. Then, it is easy to see that

$$[x^{n+1}y-y^mx,x] = x^{n+1}[y,x]-[y^m,x]x = \{[x,y^m]-x^n[x,y]\}x = 0.$$

Hence, R is commutative, by [4, Theorem].

Lemma 3. *Let R be a left s-unital ring, and let m, n be fixed non-negative integers. Suppose that R satisfies* (1). *If $(m,n) \neq (1,0)$, then $N \subseteq C$.*

Proof. If $m = 0$, then it is easy to see that R is s-unital, and R is commutative. If $m = 1$ and $n > 0$, then the assertion is clear. Thus, we assume henceforth that $m > 1$. Let $a \in N$ with $a^l = 0$, and choose t such that $m^t > l$. Then, as was shown in the proof of Lemma 2, $r^{tn}[r,a] = 0$ for any $r \in R$. Choose e such that $ea = a$ and $er = r$. Noting that $[e,a] = e^{tn}[e, a] = 0$, we obtain $(r+e)^{tn}[r, a] = (r+e)^{tn}[r+e, a] = 0$, and therefore $[r,a] = \{e^{tn}-(r+e)^{tn}\}[r,a] = \{e^{tn}-(r+e)^{tn}\}^{tn}[r,a] = 0$.

We are now ready to complete the proof of Theorem.

Proof of Theorem. (a) Lemmas 1 and 2 show that $DR = 0$. Hence, by Lemma 3, $D = RD = DR = 0$. (Note that $D \subseteq N$ by [1, Proposition 2].)

(b) Again by Lemmas 1 and 2, we have $RD = 0$. If $m \neq 1$ and $n = 0$ then it is obvious that $N \subseteq C$. On the other hand, if m > 0 and $n > 0$ then $a \in Ra$ for any $a \in N$. In fact, let $a^t = 0$, and choose $e \in R$ such that $ae = a$. Then, we have $0 = a^{tn}[a, e] = [a, e^{m^t}] = a - e^{m^t} a$, which proves that $a \in Ra$. Hence, in either case, $D = RD = 0$. (Note that $D \subseteq N$ by [1, Proposition 2].)

Remark. Let K be a field. Then, the non-commutative ring $R = \begin{pmatrix} K & 0 \\ K & 0 \end{pmatrix}$ has a right identity element and satisfies the polynomial identity $x[x, y] = 0$ (cf. Theorem (b)).

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