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SOME COMMUTATIVITY THEOREMS FOR PRIME RINGS WITH DERIVATIONS AND DIFFERENTIALLY SEMIPRIME RINGS

To the memory of Takeshi Onodera

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Throughout the present paper, R will represent a ring with center C, and U a non-zero ideal of R. Let σ, τ be ring-automorphisms of R, and set $C_{\sigma,\tau} = \{c \in R \mid c\sigma(x) = \tau(x)c \text{ for all } x \in R\}$; in particular, $C_{1,1} = C$. Given $x, y \in R$, we write $[x,y]_{\sigma,\tau} = x\sigma(y) - \tau(y)x$; in particular, $[x,y]_{1,1} = [x,y]$, in the usual sense. Let $d : x \to x'$ be a (σ,τ)-derivation of R, that is an additive map of R satisfying $(xy)' = x'\sigma(y) + \tau(x)y'$ for all $x, y \in R$. We consider the following conditions :

a) R is commutative.
b) $[u',u] = 0$ for all $u \in U$.
c) $[u',u]_{\sigma,\tau} = 0$ for all $u \in U$.
d) $[u',u]_{\sigma,\tau} \in C_{\sigma,\tau}$ for all $u \in U$.
e) U' is commutative.
f) $U'' \subseteq C$.

As a generalization of Posner's theorem [6, Theorem 2], the present authors and A. Kaya [3, Theorem 1 (2)], and independently J. H. Mayne [5, Theorem 1], have proved that if d is a non-zero ((1,1)-)derivation of a prime ring R then the conditions a) and d) are equivalent. On the other hand, L. O. Chung and J. Luh [1] have proved that if d is a non-zero derivation of a prime ring R of characteristic not 2 then the conditions a), e) for $U = R$, and f) for $U = R$ are equivalent, and more recently A. Trzepizur [7] has proved a similar result for semiprime rings.

In § 1, we generalize partially Posner's theorem in two directions (Propositions 1,2), and give a partial generalization of [3, Theorem 1 (2)] (Theorem 1). In § 2, we prove one more generalization of Posner's theorem (Theorem 2). Finally, in § 3, we generalize Trzepizur's theorem for differentially semiprime rings and the result of Chung and Luh for prime rings (Theorem 3).

1. Throughout this section, R will be a prime ring. We begin with the following partial generalization of [6, Theorem 2].

Proposition 1. *Let d be a non-zero $(\sigma, 1)$-derivation of a prime ring R. Then* a) *and* b) *are equivalent.*

Proof. Linearlizing the identity b) on U, we obtain

$$(1) \qquad [u', v] = [u, v'] \quad \text{for all } u, v \in U.$$

Replacing v by uv in (1), we get

$$[u', uv] = [u, (uv)'] = [u, u'\sigma(v) + uv'] \quad \text{for all } u, v \in U.$$

Combining this with b) and (1), we have

$$[u, uv'] = u[u, v'] = u[u', v] = [u', uv] = u'[u, \sigma(v)] + [u, uv'],$$

and therefore $u'[u, \sigma(v)] = 0$ for all $u, v \in U$, namely $u'[u, \sigma(U)] = 0$ for all $u \in U$. Noting that $\sigma(U)$ is an ideal of R, we have $u'\sigma(U)[u, v] = u'[u, \sigma(U)v] = 0$ for all $u, v \in U$. Then we can easily see that either $U' = 0$ or U is commutative. But, as is easily seen, $U' \neq 0$, and hence U is commutative. Now, it is a routine to prove that R is commutative (see [3, Lemma 1 (1)]).

Proposition 2. *Let d be a non-zero (σ, τ)-derivation of a prime ring R. Then* c) *implies* a) *and* $\sigma = \tau$.

Proof. It is easy to see that $U' \neq 0$. Linearlizing the identity c) on U, we have

$$(2) \qquad u'\sigma(v) - \tau(u)v' = \tau(v)u' - v'\sigma(u) \quad \text{for all } u, v \in U.$$

Replacing v by uv in (2), we get

$$[u', u]_{\sigma,\tau}\sigma(v) + \tau(u)[v', u]_{\sigma,\tau} - \tau(u)\tau(v)u' + u'\sigma(v)\sigma(u) = 0.$$

Combining this with c) and (2), we have

$$\begin{aligned} 0 &= -\tau(u)[u', v]_{\sigma,\tau} - \tau(u)\tau(v)u' + u'\sigma(v)\sigma(u) \\ &= -\tau(u)u'\sigma(v) + u'\sigma(v)\sigma(u) = -u'\sigma(u)\sigma(v) + u'\sigma(v)\sigma(u), \end{aligned}$$

and therefore $u'\sigma([v, u]) = 0$, namely $u'\sigma([U, u]) = 0$ for all $u \in U$. Hence, $u'\sigma(U)\sigma([x, u]) = 0$ for all $u \in U$ and $x \in R$. Since $\sigma(U)$ is a non-zero ideal of R, we see that either $U \subseteq C$ or $U' = 0$. Since $U' \neq 0$, we have $U \subseteq C$, and hence R is commutative. Thus, for any $u \in U$ we have

$0 = [u', u]_{\sigma,\tau} = u'(\sigma(u) - \tau(u))$, and so we conclude that $\sigma(u) = \tau(u)$ for all $u \in U$. By [3, Lemma 1 (2)], this proves that $\sigma = \tau$.

We conclude this section with the following partial generalization of [3, Theorem 1 (2)].

Theorem 1. *Let d be a non-zero (σ, τ)-derivation of a prime ring R of characteristic not 2. Then* c) *and* d) (*and therefore* a)) *are equivalent.*

Proof. Suppose R satisfies d). Let u be an arbitrary element of U. Then, by repeated use of d), we have

$$\begin{aligned}[(u^2)', u^2]_{\sigma,\tau} &= [u', u]_{\sigma,\tau}\sigma(u^2) - \tau(u^2)[u', u]_{\sigma,\tau} \\ &\quad + 2\tau(u)u'\sigma(u^2) - 2\tau(u^2)\tau(u)u' \\ &= 2\tau(u)\{u'\sigma(u)\sigma(u) - \tau(u)\tau(u)u'\} \\ &= 2\tau(u)\{[u', u]_{\sigma,\tau}\sigma(u) + \tau(u)[u', u]_{\sigma,\tau}\} \\ &= 4\tau(u^2)[u', u]_{\sigma,\tau}.\end{aligned}$$

Hence, $\tau(u^2)[u', u]_{\sigma,\tau} \in C_{\sigma,\tau}$, and therefore for any $x \in R$ we have

$$0 = \tau(u^2)[u', u]_{\sigma,\tau}\sigma(x) - \tau(x)\tau(u^2)[u', u]_{\sigma,\tau} = \tau([u^2, x])[u', u]_{\sigma,\tau}.$$

This proves that either $u^2 \in C$ or $[u', u]_{\sigma,\tau} = 0$. Suppose $u^2 \in C$. Then, again by d), $[u', u]_{\sigma,\tau}(\sigma(u^2) - \tau(u^2)) = 0$. If $\sigma(u^2) \neq \tau(u^2)$ then it is easy to see that $[u', u]_{\sigma,\tau} = 0$. On the other hand, if $\sigma(u^2) = \tau(u^2)$ ($\in C$) then for any $x \in R$

$$\begin{aligned}0 = ([u^2, x])' &= (u^2)'\sigma(x) + \tau(u^2)x' - x'\sigma(u^2) - \tau(x)(u^2)' \\ &= (u^2)'\sigma(x) - \tau(x)(u^2)',\end{aligned}$$

which says that $(u^2)' = u'\sigma(u) + \tau(u)u'$ is in $C_{\sigma,\tau}$. Combining this with d), we get $2\tau(u)u' \in C_{\sigma,\tau}$, and hence $\tau(u)[u', u]_{\sigma,\tau} = 0$, which implies $[u', u]_{\sigma,\tau} = 0$. We have thus shown that $[u', u]_{\sigma,\tau} = 0$ in either case.

2. In this section too, we restrict ourselves to a prime ring R with non-zero derivation $d : x \to x'$. Let $[R] = \{x \in R \mid [x', x] \in C\}$, and $(R) = \{x \in R \mid (x', x) = x'x + xx' \in C\}$. We say that d is *semicentralizing* if $R = [R] \cup (R)$. In particular, if $R = [R]$ then d is *centralizing*.

The purpose of this section is to generalize [6, Theorem 2] as follows:

Theorem 2. *If a prime ring R has a non-zero semicentralizing derivation d, then R is commutative.*

For the proof of Theorem 2, we need the following four lemmas.

Lemma 1 ([3, Lemma 2]). *Let d be semicentralizing.*

(1) *Let $x, y \in [R]$ (resp. (R)). Then $x+y \in [R]$ (resp. (R)) if and only if $x-y \in [R]$ (resp. (R)).*

(2) *If $y \in (R)$, then $[y', y^2] = [y,(y')^2] = 0$.*

Lemma 2. *Let d be semicentralizing, and let R be of characteristic not 2.*

(1) *If $y \notin [R]$, then $(y^2)' = 0$ and $(y')^4 = 0$.*

(2) *If C is not zero then d is centralizing.*

Proof. (1) Since $(y^2)' = (y', y) \in C$ and $[y', y^2] = 0$ (Lemma 1 (2)), we have

$$[(y^2+y)', y^2+y] = [(y^2-y)', y^2-y] = [y', y] \notin C,$$

which means that $y^2+y \notin [R]$ and $y^2-y \notin [R]$. Then, by Lemma 1 (1), $(y^2+y)-(y^2-y) = 2y \notin [R]$ shows that $2y^2 = (y^2+y)+(y^2-y) \in (R)$, and so $y^2 \in (R)$. Hence, $2(y^2)'y^2 = ((y^2)', y^2) \in C$, i.e., $(y^2)'y^2 \in C$. Furthermore, by Lemma 1 (2), $0 = (y^2)'[(y^2+y)',(y^2+y)^2] = 2(y^2)'[y', y^3] = 2(y^2)'y^2[y', y]$, i.e., $(y^2)'y^2[y', y] = 0$. Since $(y^2)'y^2 \in C$ and R is prime, $[y', y] \neq 0$ implies $(y^2)'y^2 = 0$. Noting here that $(y^2)' \in C$, we get

(3) $$(y', y) = (y^2)' = 0 \text{ and } (y'', y)+(y', y') = (y', y)' = 0.$$

Since $y^2+y \notin [R]$, we can apply (3) to see that $2y'y^2 = (y', y^2+y) = ((y^2+y)', y^2+y) = 0$, and so

(4) $$y'y^2 = y^2y' = 0 \text{ and } y^2y'' = (y^2y')' = 0.$$

If $y' \notin [R]$, then $(y')^2y'' = 0$ by (4). Since $[y,(y')^2] = 0$ (Lemma 1 (2)), by (3) we have $2(y')^4 = (y')^2((y'', y)+(y', y')) = 0$, i.e., $(y')^4 = 0$. Thus, we assume henceforth that $y' \in [R]$. Then, by Lemma 1 (1), either $y+y' \notin [R]$ or $y-y' \notin [R]$. We assume first that $y+y' \notin [R]$. Then, by (3) we have

(5) $$(y', y'') = (y+y', (y+y')') = 0.$$

Since $[y', y''] \in C$, (5) proves that $y'y'' \in C$ and $y''y' \in C$. Hence, by (3) and (4), we get

$$y'(y''y'+(y')^2) = (y')^2y''+(y')^3 = (y^2+(y')^2+(y', y))(y'+y'')$$

$$= (y+y')^2(y+y')' = 0.$$

Obviously, if $y''y' = 0$ then $(y')^3 = 0$. On the other hand, if $y''y' \neq 0$ then $y''y'(y'y''+(y')^2) = 0$ gives $y'y''+(y')^2 = 0$, and so $y''y'(y''+y') = 0$, whence it follows that $y''+y' = 0$. This together with (5) implies $(y')^2 = 0$. Also, in case $y-y' \notin [R]$, we can see that $(y')^3 = 0$.

(2) This is [3, Lemma 4 (3)].

Lemma 3 ([2, Lemma 1]). *Let f be a non-trivial idempotent of R. If $(f+fx-fxf)' = 0$ for all $x \in R$, then $d = 0$.*

Lemma 4. *Let Q be the Martindale quotient ring of R. Let p, q, r be elements of Q. If $puqur = 0$ for all $u \in U$, then one, at least, of p, q, r is zero.*

Proof. If x, y are elements of Q such that $xUy = 0$, then x or y is zero. By making use of this fact, we can prove the lemma in the same way as in the proof of [6, Lemma 2].

We are now ready to complete the proof of Theorem 2.

Proof of Theorem 2. By [6, Theorem 2], it suffices to show that d is centralizing, and so we may assume that R is of characteristic not 2. In view of Lemma 2 (2), we may further assume that $C = 0$. Then R satisfies the non-trivial differential identity $[x^2, x'] = 0$. By [4, Corollary 5], the central closure S of R is a primitive ring with non-zero socle. According to [2, Lemma 4], we can extend d in a unique way to a derivation of S, which is also denoted by $d : x \to x'$. Now, let e be an arbitrary idempotent in S. Then there exists a non-zero ideal A of R such that $eA \subseteq R$ and $Ae \subseteq R$. For any $a \in A$, we have either $ea(ea)' = (ea)'ea$ or $ea(ea)' = -(ea)'ea$. In either case, we have $e(ea)'ea = (ea)'ea$. Hence, we see that $(ee'-e')aea = 0$ for all $a \in A$, and so $ee' = e'$ by Lemma 4. Similarly, we can show that $e'e = e'$. We see therefore that $e' = (e^2)' = ee'+e'e = 2e'$, that is, $e' = 0$. Noting here that $f+fx-fxf$ is an idempotent for every idempotent $f \in S$ and every $x \in S$ and that d is non-zero, we see that S has no non-trivial idempotents (Lemma 3). Hence S has to be a division ring, and so R is a domain. Now, by Lemma 2 (1), we conclude that d is centralizing.

3. Throughout this section, d will represent a derivation of R, and

U a differential ideal of R with $l(U) = 0$. If R is a prime ring of characteristic not 2 and d is non-zero, then we can prove that the conditions a), e) and f) are equivalent (see Corollary 1 below).

We say that R is *differentially prime* (abbr. *d-prime*) if one of the following equivalent conditions is satisfied:

1) If I is a non-zero differential ideal of R and $xIy^{(k)} = 0$ $(x, y \in R)$ for all $k \geq 0$ then $x = 0$ or $y = 0$.

2) If I is a non-zero differential ideal of R and $x^{(k)}Iy = 0$ $(x, y \in R)$ for all $k \geq 0$ then $x = 0$ or $y = 0$.

3) If I, J are differential ideals of R and $IJ = 0$ then $I = 0$ or $J = 0$.

As is easily seen, if R is d-prime then R is either of prime characteristic or torsion free. A differential ideal P of R is said to be *d-prime* if the factor ring R/P is d-prime. The intersection of all d-prime ideals of R is called the *d-prime radical* of R. We say that R is *differentially semiprime* (abbr. *d-semiprime*) if the d-prime radical of R is zero. It is a routine to verify the equivalence of the following conditions:

i) R is d-semiprime.

ii) R contains no non-zero nilpotent differential ideals.

iii) R is differentially isomorphic to a subdirect sum of d-prime rings.

A little care is needed here. If R is d-semiprime then $l(U) = 0$ shows that the intersection of all d-prime ideals not including U is zero. Needless to say, every semiprime (resp. prime) differential ring is d-semiprime (resp. d-prime). If R is d-prime, "$l(U) = 0$" becomes "$U \neq 0$".

Lemma 5. *Suppose d is non-zero.*

(1) *If R is d-prime then $U' \neq 0$.*

(2) *If R is d-semiprime and $2R = R$ then $U'' \neq 0$.*

Proof. (1) Suppose, to the contrary, that $U' = 0$. Then, for any non-zero $u \in U$ and $x \in R$, we have $0 = (ux)' = ux'$. Hence $0 = u(yx)' = uyx'$, whence it follows that $uRx^{(k)} = 0$ for all $k \geq 1$. Hence $x' = 0$ for all $x \in R$. But this is a contradiction.

(2) It suffices to prove the case that R is d-prime. Suppose, to the contrary, that $U'' = 0$. Then $2u'v' = (uv)'' - u''v - uv'' = 0$, and hence $u'v' = 0$ for all $u, v \in U$. The relation $u'vu' = (uv)'u' = 0$ gives $u'Uu^{(k)} = 0$ for all $k \geq 1$, whence $U' = 0$ follows. This contradicts (1).

Lemma 6. *If R is d-semiprime then* e) *implies* f). *If, furthermore,*

$2R = R$, *then* e) *and* f) *are equivalent.*

Proof. It suffices to prove the case that R is d-prime. In case $U' = 0$, there is nothing to prove. We may therefore assume that $U' \neq 0$.

Since $u''[v, w'] = [u''v, w'] = [(u'v)', w'] = 0$ $(u, v, w \in U)$, we have $u''v[x, w'] = u''[vx, w'] - u''[v, w']x = 0$, and therefore $u''U[x, w']^{(k)} = 0$ for all $k \geq 0$ $(u \in U,\ x \in R)$. Hence $U'' = 0$ or $U' \subseteq C$, and so $U'' \subseteq U' \subseteq C$. Suppose now that $2R = R$. We shall show that f) implies e). Obviously, $[v', u'] = 0$ and

$$u'''[v', u'] = [u'''v', u'] = [(u'v')'', u'] - 2[u''v'', u'] - [u', u']v''' = 0.$$

Hence $u'''R[v', u']^{(k)} = 0$ for all $k \geq 0$ $(u, v \in U)$. Then, either $U''' = 0$ or U' is commutative. If $U''' = 0$ then

$$u''[v', u'] = [(uv')'', u'] - 2[u', u']v'' - [uv''', u'] = 0,$$

and hence $u''R[v', u']^{(k)} = 0$ for all $k \geq 0$. Noting here that $U'' \neq 0$ by Lemma 5 (2), we get e), again.

Careful scrutiny of the proof of Proposition 1 shows the following

Lemma 7. *Let R be a d-prime ring, and $d \neq 0$. Then* b) *implies* a).

We are now ready to prove the following principal theorem of this section.

Theorem 3. *Let R be a d-semiprime ring with $2R = R$. If $K = \{x \in R \mid x' = 0\}$ is commutative then the conditions* a), e) *and* f) *are equivalent.*

Proof. In view of Lemma 6, it remains only to prove that e) implies a).

We claim first that $U' \subseteq C$. To see this, we may assume that R is d-prime. As was shown in the proof of Lemma 6, either $U'' = 0$ or $U' \subseteq C$. If $U'' = 0$ then $U' = 0$ by Lemma 5 (2), and therefore $U' \subseteq C$ in either case.

Now, let $\bigcap_{\lambda \in \Lambda} P_\lambda = 0$ with d-prime ideals $P_\lambda \not\supseteq U$. Put $\Lambda_1 = \{\lambda \in \Lambda \mid P_\lambda \supseteq U'\}$ and $\Lambda_2 = \{\lambda \in \Lambda \mid P_\lambda \not\supseteq U'\}$. Let D be the commutator ideal of R. Then, Lemma 7 shows that $D \subseteq P_\lambda$ for all $\lambda \in \Lambda_2$. Hence $D'U \subseteq (DU)' + DU' \subseteq P_\lambda$ for all $\lambda \in \Lambda$, and therefore $D' \subseteq \bigcap_{\lambda \in \Lambda} P_\lambda = 0$, namely $D \subseteq K$. By hypothesis, D is then a commutative ideal. Now, let $\mu \in \Lambda_1$. Then $\bar{R} = R/P_\mu$ is a prime ring. (Note that $R'U \subseteq (RU)' + RU' \subseteq P_\mu$

implies $R' \subseteq P_\mu$.) If $D \nsubseteq P_\mu$ then $\bar{D}$ is a non-zero commutative ideal of the prime ring $\bar{R}$. Hence, by [3, Lemma 1 (1)], $\bar{R}$ is commutative, which contradicts $\bar{D} \neq 0$. We have thus seen that $D \subseteq P_\lambda$ for all $\lambda \in \Lambda$, namely $D = 0$, which proves the commutativity of R.

Careful scrutiny of the proof of Theorem 3 shows the following

Corollary 1. *Let R be a prime ring of characteristic not* 2. *If $d \neq 0$ or K is commutative then the conditions* a), e) *and* f) *are equivalent.*

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