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SOME H-SEPARABLE POLYNOMIALS OF DEGREE 2

To the memory of Takeshi Onodera

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Throughout, B will mean a non-commutative ring with identity element 1, and $B[X; \rho]$ will be a skew polyonmial ring $\sum_{i=0}^{\infty} X^i B$ whose multiplication is given by $\alpha X = X\rho(\alpha)\ (\alpha \in B)$ where ρ is an automorphism of B. A polynomial $f \in B[X; \rho]$ will be called to be *separable* (resp. *H-separable*) if f is monic, $fB[X; \rho] = B[X; \rho]f$, and the factor ring $B[X; \rho]/B[X; \rho]f$ is a separable (resp. H-separable) extension of B in the sense of Y. Miyashita [5, p.115] (resp. of K. Sugano [10, p.29]).

Now, in [9], we called a separable polynomial $f \in B[X; \rho]$ to be *of Galois type* when $A = B[X; \rho]/B[X; \rho]f$ is imbedded in a G-Galois extension N of B (in the sense of [5, p.116]) such that $A = N^H (= \{c \in N; \tau(c) = c$ for all $\tau \in H\})$ for $H = \{\sigma \in G; \sigma|A$ (the restriction of σ to A) $= 1\}$; and proved that a separable polynomial $f = X^2 - Xa - b \in B[X; \rho]$ is of Galois type if and only if $\delta(f) = a^2 + 4b$ is inversible in B.

In this note, as to the case of non-Galois type, we shall study the separable polynomial of degree 2 in $B[X; \rho]$ whose discriminants are contained in $J(B)$, the Jacobson radical of B, and shall prove that such polynomials are characterized as H-separable polynomials of degree 2 in $B[X; \rho]$ with $2 \in J(B)$; this is an extension of [9, Cor. 2.2] to the case that B is non-commutative, which is a sample made to classify non-commutative separable extensions.

Throughout this paper, we denote the center of B by Z. As to the other notations and terminologies, we follows the previous ones [6]—[7].

Now, let $f = X^2 - Xa - b$ be a separable polynomial in $B[X; \rho]$. Then, by [7, p.168, (i), (ii) and Th. 1], we have

(i) $$\alpha a = a\rho(\alpha), \quad \alpha b = b\rho^2(\alpha)$$

for all $\alpha \in B$,

(ii) $$\rho(a) = a, \quad \rho(b) = b.$$

By (ii), f is a separable polynomial in $B[X; \rho]_{(2)}$ in the sense of [6, p.65]. Hence by [6, Lemma 2.1, (2, xvii), Lemma 2.2, (2, xix)], there are elements b_1, $b_2 \in B$ and $b_4 \in Z$ such that

(iii) $$1 = b_4+\rho(b_4)+ab_2 = b(b_1+\rho(b_1))-ab_2$$

(iv) $$4 = \delta(f)(b_1+\rho(b_1))$$

(v) $$a = \delta(f)(b_1b_4-b_2^2)a.$$

The above remarks (i−v) will play important rôles in the subsequent considerations. Our first result is the next theorem which is closely connected with our previous one [9, Th. 2.1].

Theorem 1. *Let* $f = X^2 - Xa - b \in B[X; \rho]$, *and* $fB[X; \rho] = B[X; \rho]f$. *Then,* f *is a separable polynomial whose discriminant is contained in* $J(B)$ *if and only if*

(1) $2 \in J(B)$, $a = 0$, $b \in B^{\rho}$ *and is inversible,*

(2) Z/Z^{ρ} *is a* $(\rho|Z)$*-Galois extension of rank* 2

where $(\rho|Z) = \{1, \rho|Z\}$.

Proof. we assume (1) and (2). Then, by (2) and [1, Lemma 1.6], there is an element c in Z such that $c+\rho(c) = 1$. Hence, it follows from (1) and [6, Lemma 2.3] that f is separable and $\delta(f) \in J(B)$. As to the converse, we assume that f is separable and $\delta(f) \in J(B)$. Let b_1, b_2 and b_4 be as in (iii−v). Then, by (iv) and (v), 2^2 and a are contained in $J(B)$. Since the radical of the factor ring $B/J(B)$ is $J(B)$, if follows that $2 \in J(B)$. Noting $ab_2 \in J(B)$, we see that $1+ab_2 = b(b_1+\rho(b_1))$ is inversible in B and so is b. Moreover, by(iii), we have

$$1-ab_2-2\rho(b_4) = b_4-\rho(b_4)$$

which is inversible in B. Since $b_4 \in Z$, it follows from (i) that $a(b_4-\rho(b_4)) = 0$, which implies $a = 0$ and $1 = b(b_1+\rho(b_1))$ (iii). Further, noting (i), we have that for any $\alpha \in Z$,

$$\rho^2(\alpha) = \rho^2(\alpha)b(b_1+\rho(b_1)) = \alpha b(b_1+\rho(b_1)) = \alpha.$$

This shows $\rho^2|Z = 1$ (cf. [2, Prop. 3.2] and [8, Lemma 4]). Since $b_4-\rho(b_4) \neq 0$ and $b_4 \in Z$, it follows that $\rho|Z$ is of order 2. we set here $r = b_4-\rho(b_4)$. Then

$$1 = r^{-1}b_4-r^{-1}\rho(b_4) \text{ and } 0 = r^{-1}b_4-r^{-1}\rho(\rho(b_4)).$$

Hence we see that $\{(r^{-1}b_4, 1), (r^{-1}, \rho(b_4))\}$ is a $(\rho|Z)$-Galois coordinate system for Z/Z^{ρ}. Thus, Z/Z^{ρ} is a $(\rho|Z)$-Galois extension of rank 2 in the senses of [1, Def. 1.4] and [5, p.116]. This completes the proof.

Now, in virtue of Th. 1 and S. Ikehata [3] and [4], we can prove the next which is an extension of [9, Cor. 2.2] to non-commutative case, and this is our main result.

Theorem 2. *Let $f \in B[X; \rho]$, and $\deg f = 2$. Then, f is a separable polynomial in $B[X; \rho]$ whose discriminant is contained in $J(B)$ if and only if f is an H-separable polynomial in $B[X; \rho]$ with $2 \in J(B)$.*

Proof. If f is a separable polynomial with $\delta(f) \in J(B)$ then $2 \in J(B)$ and f is H-separable by the results of Th. 1 and [3, Prop. 1. 4]. To see the converse, we set $f = X^2 - Xa - b$, and assume that $2 \in J(B)$ and f is H-separable. Then, by [4, Lemma 1], we have $a = 0$, that is, $f = X^2 - b$. As is well known, any H-separable extension is separable (cf. [10, p. 29]). Hence f is a separable polynomial in $B[X; \rho]$ with $\delta(f) = 4b \in J(B)$, completing the proof.

Remark. Combining Th. 2 with S. Ikehata [3, Th. 2.1], we obtain the result of [9, Th. 2.1], which is as follows: Let B be commutative, and f a monic polynomial of degree 2 in $B[X; \rho]$ with $fB[X; \rho] = B[X; \rho]f$. Then. f is separable and $\delta(f) \in J(B)$ if and only if $B[X; \rho]/B[X; \rho]f$ is an Azumaya B^{ρ}-algebra with $2 \in J(B)$.

Example. Let Q be the field of rational numbers, and $B = Q[x, y]$ the ring of polynomials of indeterminates x, y with coefficients in Q where x, y are independent, $\alpha x = x\alpha$ and $\alpha y = y\alpha$ ($\alpha \in Q$). Moreover, let ρ be the mapping of B into itself defined by $g(x, y) \to g(y, x)$. Then, ρ is an automorphism of B of order 2. we set here $C = B^{\rho}$. Since $x+y$, $xy \in C$, it follows that $h(X) = X^2 - (x+y)X + xy \in C[X]$ and $h(x) = 0$. Hence $B = C + Cx \neq C$. As is easily seen, we have $(x-y)B \neq B$, that is, $x-y$ $(= x - \rho(x))$ is not inversible in B. Hence, it follows from [6, Lemma 1.2] that B is not (ρ)-Galois over C. Now, we consider the skew polynomial ring $B[X; \rho]$ and polynomial $f = X^2 - 1 \in B[X; \rho]$. Then, $fB[X; \rho] = B[X; \rho]f$, and f is not H-separable by [3, Th. 2.1]. However, since $\delta(f) = 4$ is inversible in B, it is seen by [7, Th. 2] that f is Galois and separable. Thus, f is a separable polynomial of Galois type which is not H-separable. In case $B = Q(x, y)$(the quotient of $Q[x, y]$), $X^2 - 1 \in B[X; \rho]$ is a separable polynomial of Galois type which is H-separable (cf. [3, Prop. 1.4].

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