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ON CONNECTEDNESS OF STRONGLY ABELIAN EXTENSIONS OF RINGS

Dedicated to Prof. Noboru Itô on his 60th birthday

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A ring with an identity 1 is said to be a connected ring if 1 and 0 are only idempotents of its center. Recently in [1], we have studied on the connectedness of p-Galois extensions of connected rings of characteristic $p>0$. In this paper, we shall continue the study on the connectedness of strongly abelian extensions of connected rings.

As is proved in [3], a strongly abelian extension of a ring is obtained by a homomorphic image of a skew polynomial ring of automorphism type, and so as a preparation, some general remarks about a skew polynomial ring of automorhphism type are given in § 1.

In § 2, we study necessary and sufficient conditions for a strongly cyclic extension over a connected ring to be connected, and results of this section are applied in § 3 for a study on the connectedness of some types of strongly abelian extensions.

1. Notations and general remarks. Throughout this paper, we assume that A is a ring with an identity 1 such that $n(>1)$ is an invertible element for some integer n and the center $C(A)$ contains a primitive n-th root ζ of 1 such that $\{1-\zeta^i\,;\ i=1,\dots,n-1\}$ are invertible elements.

Let $\{\rho_i\,;\ i=1,\dots,m\}$ be automorphisms of A and $\mathscr{A}=\{a_{ij}\in U(A)\,;\ i,j=1,\dots,m\}$ where $U(A)$ is the group of all invertible elements of A. If $\{\rho_i\,;\ i=1,\dots,m\}$ and $\mathscr{A}$ satisfy the conditions

(1) $a_{ij}a_{ji}=a_{ii}=1$

(2) $\rho_i\rho_j\rho_i^{-1}\rho_j^{-1}=\tilde{a}_{ij}$, the inner automorphism $(a_{ij})_l(a_{ji})_r$

(3) $a_{ij}\rho_j(a_{ik})a_{jk}=\rho_i(a_{jk})a_{ik}\rho_k(a_{ij})$,

for all $i,j,k=1,\dots,m$, then the set of all polynomials $\{\sum X_1^{\nu_1}X_2^{\nu_2}\cdots X_m^{\nu_m}a_{\nu_1\nu_2\cdots\nu_m}\,;\ a_{\nu_1\nu_2\cdots\nu_m}\in A\}$ becomes an associative ring by the rules

$$aX_i=X_i\rho_i(a) \text{ for all } a\in A \text{ and}$$
$$X_iX_j=X_jX_ia_{ij}. \text{ (See [3]).}$$

This ring is denoted by $R_m=A[X_1,\dots,X_m\,;\ \rho_1,\dots,\rho_m,\mathscr{A}]$ or $R_m=$

$A[X_1,\dots,X_m\,;\,\rho_1,\dots,\rho_m,\{a_{ij}\,;\,i,j=1,\dots,m\}]$ and is called a skew polynomial ring of automorphism type. Moreover, by R_k $(0\leqq k\leqq m)$, we denote the skew polynomial ring $A[X_1,\dots,X_k\,;\,\rho_1,\dots,\rho_k,\{a_{ij}\,;\,i,j=1,\dots,k\}]$ which is a subring of R_m, where $R_0=A$. In particular, if $m=1$, we denote it by

$$R=A[X\,;\,\rho]=\{\sum X^i a_i\,;\,a_i\in A\}$$

and its multiplication is given by

$$aX=X\rho(a)\ \text{for}\ a\in A.$$

Remark 1.1. For a permutation π of k letters, $1,2,\dots,k$, $(k\leqq m)$, we have an A-ring isomorphism $R_k\cong A[X_{\pi(1)},\dots,X_{\pi(k)}\,;\,\rho_{\pi(1)},\dots,\rho_{\pi(k)},\{a_{\pi(i)\pi(j)}\,;\,i,j=1,\dots,k\}]$ which maps X_i to $X_{\pi(i)}$ $(i=1,\dots,k)$.

Remark 1.2. ρ_{k+1} can be extended to an automorphism ρ^*_{k+1} of R_k by $\rho^*_{k+1}(X_j)=X_ja_{jk+1}$ for $j=1,\dots,k$ and $\rho^*_{k+1}|A=\rho_{k+1}$. Moreover, there holds $R_{k+1}\cong R_k[X_{k+1}\,;\,\rho^*_{k+1}]$.

Definition 1.3. Let $g=X_i^s+\sum_{h=0}^{t-1}f_h(X_1,\dots,X_{i-1},X_{i+1},\dots,X_m)\in R_m$. g is said to be a generator in R_m if $s\geqq 1$ and $gR_m=R_mg$. A generator g in $R=A[X\,;\,\rho]$ is said to be weakly irreducible (abbreviate w-irreducible) if g has no proper factors which are generators.

Let $G=(\sigma_1)\times(\sigma_2)\times\cdots\times(\sigma_m)$ be an abelian group such that $|\sigma_i|=n_i$ and $n=\prod_{i=1}^m n_i$.

Definition 1.4. A G-Galois extension B of A is said to be a G-strongly abelian extension if $B_A\oplus>A_A$(i.e., A_A is an A-direct summand of B_A) and there exist $x_1,\dots,x_m\in U(B)$ such that $\sigma_i(x_j)=x_j(\zeta_j)^\varepsilon$ where $\zeta_j=\zeta^{n/n_j}$ and $\varepsilon=\delta_{ij}$, the Kronecker's delta.

Remark 1.5. A has a G-stronly abelian extension if and only if there exist automorphisms $\{\rho_i\,;\,i=1,\dots,m\}$ of A and a set of elements $\mathscr{A}=\{a_{ij}\in U(A)\,;\,i,j=1,\dots,m\}$ which satisfy conditions (1)$-$(3), $\rho_i(\zeta)=\zeta$ and there exist elements $\alpha_1,\dots,\alpha_m\in U(A)$ such that $X_k^{n_k}-\alpha_k$ is a generator in $R_m=A[X_1,\dots,X_m\,;\,\rho_1,\dots,\rho_m,\mathscr{A}]$ for $k=1,\dots,m$. Moreover, if this is the case, B is isomorphic to R_m/M, $M=(X_1^{n_1}-\alpha_1,\dots,X_m^{n_m}-\alpha_m)R_m$ and $\sigma_i(x_j)=x_j(\zeta_j)^\varepsilon$ where x_j is the coset of X_j modulo M. Hence, we may write

$$B = A[x_1, \dots, x_m \,;\, \rho_1, \dots, \rho_m, \mathscr{A}\,] \\ = \sum \oplus (x_1^{\nu_1} x_2^{\nu_2} \cdots x_m^{\nu_m})A \quad (0 \leqq \nu_i \lneqq n_i - 1)$$

with $x_i^{n_i} = \alpha_i \in U(A)$, $ax_i = x_i\rho_i(a)$ for $a \in A$, $\rho_i(\zeta) = \zeta$ and $x_ix_j = x_jx_ia_{ij}$.

Remark 1.6. If $\mathscr{A} = \{1\}$, we denote R_m by $A[X_1, \dots, X_m \,;\, \rho_1, \dots, \rho_m]$. Then $X_i^{n_i} - \alpha_i$ is a generator in R_m if and only if $\alpha_i \in \bigcap_{j=1}^{m} A^{\rho_j}$ where $A^{\rho_j} = \{a \in A \,;\, \rho_j(a) = a\}$.

Remark 1.7. If $G = (\sigma)$ is a cyclic group of order n then we say that a G-strongly abelian extension B of A is an n-strongly cyclic extension. In this case, B is obtained by $A[x \,;\, \rho] = \sum_{i=0}^{n-1} \oplus x^iA$ with $x^n = \alpha \in U(A)$, $ax = x\rho(a)$ for $a \in A$, $\rho(\zeta) = \zeta$ and $\sigma(x) = x\zeta$.

Remark 1.8. Let $f(X) = X^s + \sum_{i=0}^{s-1} X^i a_i \in R = A[X \,;\, \rho]$. If $a_0 \in U(A)$ then $f(X)$ is a generator in R if and only if $a_i \in A^{\rho}$ and $\rho^i(a)a_i = a_i\rho^s(a)$ for any $a \in A$ and $i = 0, 1, \dots, s$. Hence $f(X)$ is contained in $C(A^{\rho})[X]$.

Let H be a group. A normal subgroup N of H is said to be a small group (abbreviate an s-group) if only subgroup H' of H such that $H = NH'$ is H. The followings are proved in [1].

Remark 1.9. Let A be connected and let B/A be an H-Galois extension for a finite group H.

(1) If B is disconnected, then there exists a nontrivial idempotent $e \in C(B)$ such that $e\tau(e) = 0$ or $\tau(e) = e$ for every $\tau \in H$.

(2) If B^N is connected for an s-subgroup N of H, then B is connected.

2. Connected n-strongly cyclic extensions. The purpose of this section is to study about the connectedness of an n-strongly cyclic extension of a connected ring. For this, we denote by B an n-strongly cyclic extension of A. Thus we may assume that $B = A[x \,;\, \rho] = \sum_{i=0}^{n-1} \oplus x^iA$ for some automorphism ρ of A and an element $\alpha \in U(A)$ such that $\rho(\zeta) = \zeta$, $x^n = \alpha$, $\rho^n = \tilde{\alpha}^{-1}$, $ax = x\rho(a)$ for $a \in A$ and $\sigma(x) = x\zeta$.

Theorem 2.1. *Let A be connected. Then B is connected if and only if $f(X) = X^n - \alpha$ is w-irreducible.*

Proof. Let B be connected. If $f(X)$ is not w-irreducible then $f(X) = g(X)h(X)$ for some generators $g(X) = \sum_{i=0}^{s} X^i a_i$ and $h(X) = \sum_{i=0}^{t} X^i b_i$. Since both a_0 and b_0 are invertible elements, $g(X)$ and $h(X)$ are contained in $C(A^{\rho})[X]$ by Remark 1.8. Hence'

$$nX^{n-1} = f'(X) = g'(X)h(X) + g(X)h'(X)$$

for an ordinal derivation ' of $C(A^{\rho})[X]$. Since $nx^{n-1} = g'(x)h(x) + g(x)h'(x)$ is an invertible element in $C(A^{\rho})[x]$ which is a subring of B, $(g(x))$ and $(h(x))$ are co-maximal ideals such that $0 = (g(x))(h(x)) = (g(x)) \cap (h(x))$. Thus $B = B/(g(x)h(x)) \cong B/(g(x)) \oplus B/(h(x))$ and this is a contradiction.

Conversely, assume $f(X)$ is w-irreducible and B is disconnected. Then there exists a nontrivial idempotent $e \in C(B)$ such that either $e\tau(e) = 0$ or $\tau(e) = e$ for $\tau \in (\sigma)$ by Remark 1.9. Let $H = \{\tau \in (\sigma) ; \tau(e) = e\} = (\sigma^m)$. Then $T_\sigma(e ; m) = e + \sigma(e) + \cdots + \sigma^{m-1}(e)$ is an idempotent of $C(A)$ and so $T_\sigma(e ; m)$ is either 0 or 1. If $T_\sigma(e ; m) = 0$ then we have a contradiction that $0 = eT_\sigma(e ; m) = e$. Thus $T_\sigma(e ; m) = 1$. Let $T = B^{\sigma^m}$. Then $T = \sum_{i=0}^{m-1} \oplus (x^{m'})^i A$ where $m' = |H|$ and $m = n/m'$. Further we can easily see that $T^\sigma = A$ and $\{x_i = y_i = \sigma^i(e) ; i = 0, 1, \ldots, m-1\}$ satisfies $\sum_{i=0}^{m-1} x_i \sigma^j(y_i) = \delta_{1,\sigma^j}$ for $j = 1, \ldots, m$. Thus T/A is a $\sigma | T$-cyclic extension and, by ([6, Theorem 2.3]),

$T = Ae + A\sigma(e) + \cdots + A\sigma^{m-1}(e)$ and this sum is a direct sum since $\sigma^i(e)\sigma^j(e) = 0$ for each $i, j = 0, 1, \ldots, m-1$ with $i \neq j$.

(1) Let $y = a_0 e + a_1\sigma(e) + \cdots + a_{m-1}\sigma^{m-1}(e)$ where $y = x^{m'}$ and $a_i \in A$.

Then

(2) $y = x^{-1}yx = \rho(a_0)e + \rho(a_1)\sigma(e) + \cdots + \rho(a_{m-1})\sigma^{m-1}(e)$

and

(3) $ay = y\sigma^{m'}(a) = aa_0 e + aa_1\sigma(e) + \cdots + aa_{m-1}\sigma^{m-1}(e)$
$= a_0\rho^{m'}(a)e + a_1\rho^{m'}(a)\sigma(e) + \cdots + a_{m-1}\rho^{m'}(a)\sigma^{m-1}(e)$

for any $a \in A$. Thus we obtain

(4) $y\sigma^i(e) = a_i\sigma^i(e) = \rho(a_i)\sigma^i(e)$ for $i = 0, 1, \ldots, m-1$

by (1) and (2), and

(5) $aa_i\sigma^i(e) = a_i\rho^{m'}(a)\sigma^i(e)$ for $a \in A$ by (3).

Noting that $T_\sigma(\sigma^i(e) ; m) = 1$, we have

(6) $\rho(a_i) = T_\sigma(\rho(a_i)\sigma^i(e) ; m) = T_\sigma(a_i\sigma^i(e) ; m) = a_i$

by (4) and

$$(7)\quad aa_i = T_\sigma(aa_i\rho^i(e)\,;\,m) = T_\sigma(a_i\rho^{m'}(a)\sigma^i(e)\,;\,m) = a_i\rho^{m'}(a).$$

Further

$$\alpha\sigma^i(e) = y^m\sigma^i(e) = (y\sigma^i(e))^m = (a_i\sigma^i(e))^m = a_i^m\sigma^i(e) \text{ (by (4))}$$

and hence, $\alpha = T_\sigma(\alpha\sigma^i(e)\,;\,m) = T_\sigma(a_i^m\sigma^i(e)\,;\,m) = a_i^m$.

Therefore

$$\begin{aligned} X^n-\alpha &= (X^{m'})^m-(a_i)^m \\ &= (X^{m'}-a_i)((X^{m'})^{m-1}+(X^{m'})^{m-2}a_i+\cdots+(a_i)^{m-1}) \end{aligned}$$

by (6). Moreover, (6) and (7) show that $X^{m'}-a_i$ and $X^{m'}+(X^{m'})^{m-1}a_i+\cdots+(a_i)^{m-1}$ are generators. This is a contradiction.

We say that t is the index of ρ and denote it by *ind.*ρ if t is the index of the subgroup of inner automorphims in (ρ). Since $\rho^n = \tilde{\alpha}^{-1}$, *ind.*$\rho \leqq n$.

Lemma 2.2. (i) *If ind.*$\rho = n$ *then* $X^n-\alpha$ *is* w*-irreducible.*

(ii) *If ind.*$\rho = 1$ *then we may assume* $R = A[X\,;\,\rho] = A[Y]$, *a polynomial ring with a commutative indeterminate* $Y \in R$ *and* $X^n-\alpha = (Y^n-z)a^n$ *for some central polynomial* Y^n-z *and* $a \in U(A)$.

Proof. (i) If a generator $g(X)$ is a factor of $X^n-\alpha$, then the constant term a_0 of $g(X)$ must be an invertible element and $\rho^k = \tilde{a}_0^{-1}$ for $k = deg\ g(X)$. Thus $k = n$ since *ind.*$\rho = n$.

(ii) Let $\rho = \tilde{a}^{-1}$ for some $a \in U(A)$. Then $\tilde{\alpha}^{-1} = \rho^n = \tilde{a}^{-n}$ implies $\alpha = a^n z$ for some $z \in U(C(A))(= U(C(A))^\rho)$. Then $Y = Xa^{-1}$ is central in $A[X\,;\,\rho]$ and $A[X\,;\,\rho] = A[Y]$. Further $X^n-\alpha = ((Xa^{-1})^n-z)a^n = (Y^n-z)a^n$ for a central polynomial Y^n-z.

Corollary 2.3. *Let* A *be connected and* n *a prime. Then* $X^n-\alpha$ *is either* w*-irreducible or a product of generators of degree* 1.

Proof. *Ind.*ρ is either n or 1. If *ind.*$\rho = n$ then $X^n-\alpha$ is w-irreducible by Lemma 2.2.(i). While, if *ind.*$\rho = 1$, then $R = A[Y]$ and $X^n-\alpha = (Y^n-z)a^n$ for $Y = Xa^{-1}$ with $a \in U(A)$ by Lemma 2.2.(ii). Hence, if $X^n-\alpha$ is not w-irreducible, then Y^n-z is reducible in $C(R) = C(A)[Y]$ and so a product of linear factor $\Pi_{i=1}^n(Y-u\zeta^i)$ by ([5, Lemma 1.4]). Hence $X^n-\alpha = \Pi_{i=1}^n(Y-u\zeta^i)a^n = \Pi_{i=1}^n(Ya-ua\zeta^i) = \Pi_{i=1}^n(X-ua\zeta^i)$ and each $X-ua\zeta^i$ is a generator.

Let $U(A)_n^\rho = \{a \in U(A)^\rho\,;\,\rho^n = \tilde{a}\}$. Then we have the following

Theorem 2.4. *Let A be connected and n a prime. Then A has a connected n-strongly cyclic extension if and only if one of the following conditions* (a) *and* (b) *is satisfied.*

(a) $n \leqq (U(C(A)) : U(C(A))^n)$, *the index of the subgroup* $U(C(A))^n = \{c^n ; c \in U(C(A))\}$.

(b) *A has an automorphism ρ of index n such that $\rho(\zeta) = \zeta$ and* $U(A)_n^\rho \neq \phi$.

Proof. First, we assume that A has a connected n-strongly cyclic extension B. Then there exist an automorphism ρ of A and an element $\alpha \in U(A)$ such that $X^n - \alpha$ is w-irreducible in $R = A[X; \rho]$ and $B \cong R/(X^n - \alpha)$. If it is possible to choose ρ as inner, then we may assume $R = A[Y]$ and $B \cong R/(Y^n - z)$ for some central polynomial $Y^n - z$ which is irreducible in $C(R) = C(A)[Y]$ by Corollary 2.3. Hence $z \in U(C(A)) \setminus U(C(A))^n$. Since $z\zeta^i U(C(A))^n$ $(i = 0, 1, \dots, n-1)$ are distinct cosets in $U(C(A))/U(C(A))^n$, there holds (a). On the other hand, if ρ is non inner, then $ind.\rho = n$ and $\alpha \in U(A)_n^\rho$. Conversely, if (a) is hold then $C(A)$ has a commutative connected n-strongly cyclic extension Z and $B = Z \otimes_{C(A)} A$ is a connected n-strongly cyclic extension of A. On the other hand, if (b) is hold then $X^n - \alpha$ is w-irreducible for $\alpha \in U(A)_n^\rho$ by Lemma 2.2.(i) and $B = A[X; \rho]/(X^n - \alpha)$ is connected by Theorem 2.1.

Let $n = p_1^{e_1} p_2^{e_2} \cdots p_s^{e_s}$ and $\tau = \sigma^{p_1 p_2 \cdots p_s}$ where $p_1, \dots, p_s$ are distinct primes. Then (τ) is an s-subgroup of (σ). Hence if A is connected then B is connected if B^τ is connected by Remark 1.9. Therefore we may assume that $n = p_1 p_2 \cdots p_s$ to study the connectedness of B over a connected ring A. Thus in the following we assume that $n = p_1 p_2 \cdots p_s$, a product of distinct primes. Let $\tau_i = \sigma^{p_i}$ and $B_i = B^{\tau_i}$. Then $B_i = \sum_{j=0}^{p_i - 1} \oplus (y_i)^j A \cong A[Y; \rho^{q_i}]/(Y^{p_i} - \alpha)$ where $y_i = x^{q_i}$ and $q_i = n/p_i$.

Theorem 2.5. *Let A be connected. Then B is connected if each B_i is connected. Conversely, if B is connected and $U(A)^{\rho^{q_i}} = U(A)^\rho$ for all i, then B_i is connected for all i.*

Proof. Let B be connected and $U(A)^{\rho^{q_i}} = U(A)^\rho$ for all i. An element $c = \sum_{j=0}^{p_i - 1} (y_j)^j a_j \in B_i$ is contained in $C(B_i)$ if and only if $\rho^{q_i}(a_j) = a_j$ and $(\rho^{q_i})^j(a) a_j = a_j a$ for any $a \in A$. But this means that c is also contained in $C(B)$. Thus each B_i is connected. Conversely, assume that each B_i is connected and we put $S_1 = B^{\theta_1}$ where $\theta_1 = \sigma^{q_1}$. First, we show that B is connected if B_1 and S_1 are connected. If this assertion is true then we can reduce the connectedness of S_1 from that of $B_i, i = 2, \ldots, s$ applying the same methods on S_1. Suppose now B is disconnected. Then there exists a non-trivial idempotent $e \in C(B)$ such that $e\tau_1(e) = 0$ and $e\theta_1(e) = 0$ by Remark 1.9. We now show that $\theta_1^i(e)\theta_1^j(e) = 0$ for $i \neq j$. For, we may assume $i < j$ and $1 \leqq j-i \leqq p_1-1$. Hence θ_1^{j-i} is a generator of (θ_1). Hence, if $\theta_1^i(e)\theta_1^j(e) \neq 0$, then $\theta_1^i(e\theta_1^{j-i}(e)) \neq 0$ and this implies a contradiction $e = \theta_1^{j-i}(e) \in S_1 = B^{\theta_1}$. Therefore

$$f = e + \theta_1(e) + \cdots + \theta_1^{p_1-1}(e) = 1$$

since f is a central idempotent of S_1. Next we shall show that $\tau_1(e)\theta_1^i(e) = 0$ for $i = 0, 1, \ldots, p_1-1$. For, $\tau_1(e)\theta_1^i(e) = \sigma^{p_1}(e\sigma^{-p_1}\theta_1^i(e)) = \sigma^{p_1}(e\sigma^{q_1 i - p_1}(e))$ and $(q_1^i - p_1, p_j) = 1$ for $j = 1, \ldots, s$. Therefore we have a contradiction $e = \sigma^{q_1 i - p_1}(e) \in B^{\sigma} = A$ if $\tau_1(e)\theta_1^i(e) \neq 0$. Thus we have a contradiction $\tau_1(e) = \tau_1(e)f = 0$ again.

Corollary 2.6. *Let A be connected. Then A has a connected n-strongly cyclic extension B if there exist an automorphism ρ of A and an element $\alpha \in U(A)_n^{\rho}$ which satisfy*

(1) $\rho(\zeta) = \zeta$

(2) *$ind.\rho^{q_i} = p_i$ for $i = 1, \ldots, k$*

(3) *$ind.\rho^{q_i} = 1$ and $\alpha(\zeta^{q_i})^j U(C(A))^{p_i}$ $(j = 0, 1, \ldots, p_i-1)$ are distinct cosets in $(U(C(A)) : U(C(A))^{p_i})$ for $i = k+1, \ldots, s$.*

Proof. If there exist an automorphism ρ and an element $\alpha \in U(A)_n^{\rho}$ which satisfy conditions (1)–(3), then $B = A[X; \rho]/(X^n - \alpha)$ is an n-strongly cyclic extension. Further (2) and (3) show that each B_i is connected, and so B is connected.

If A is a connected commutative ring and B is a commutative n-cyclic extension, then B is an n-strongly cyclic extension, and so B is connected if and only if there exists an element $\alpha \in U(A)$ such that $\alpha \notin U(A)^{p_i}$ for each i. Moreover, it is known that if A is a local ring (resp. a domian) then so is B, and conversely (See [4, § 1]). Hence we have the following

Corollary 2.7. *Let A be a connected commutative ring. Then A has a connected commutative n-cyclic extension B if and only if there exists $\alpha \in U(A)$ such that $\alpha \notin U(A)^{p_i}$ for $i = 1, \ldots, s$. Further, if this is the case, B is a local ring (resp. a domain) if and only if so is A.*

Let A be a two sided simple ring. Then each ideal of $A[X\,;\,\rho]$ is generated by a generator and the ideal is maximal if and only if the generator is w-irreducible. Since each generator $f(X)$ is decomposed into $\Pi_{i=1}^{k} g_i(X)$, a product of w-irreducible polynomials, if $g_i(X) \neq g_j(X)$ for $i \neq j$ then $(f(X))$ is a product ($=$ the intersection) of the maximal ideals $(g_i(X))$. Hence we have the following

Theorem 2.8. *Let A be a two sided simple ring (resp. a simple artinian ring).*

(a) $B = A[X\,;\,\rho]/(X^n - \alpha)$ *is a finite direct sum of two sided simple rings (resp. simple artinian rings).*

(b) $B = A[X\,;\,\rho]/(X^n - \alpha)$ *is a two sided simple ring (resp. a simple artinian ring) if and only if B is connected.*

Proof. (a) Let $X^n - \alpha = \Pi_{i=1}^{k} g_i(X)$ be a decomposition into w-irreducible polynomials in $A[X\,;\,\rho]$. Since $X^n - \alpha$ is separable in R, $(g_i(X))$ and $(g_j(X))$ are co-maximal ideals if $i \neq j$ by ([7, Theorem 1.10]). Thus $B = \sum_{i=1}^{k} \oplus A[X\,;\,\rho]/(g_i(X))$ and each $A[X\,;\,\rho]/(g_i(X))$ is a two sided simple ring (resp. a simple artinian ring).

(b) is an immediate consequence of (a).

As is shown in ([1, Lemma 2.1]), if A is of characteristic p for a prime p and B is a connected p^e-cyclic extension of A then A is also connected. But the following example shows that there exists a connected n-strongly cyclic extension B of A even if A is disconnected.

Example 2.9. Let $A = Q \oplus Q$ be the direct sum of 2-copies of the rational numbers field Q. Then $C(A) = A$ is disconnected. The map $\rho : A \to A$ such that $(q_1, q_2) \to (q_2, q_1)$ is an automorphism of A of order 2, $A^{\rho} = Q = \{(q, q)\,;\, q \in Q\}$ and $X^2 - 2$ is a generator of $R = A[X\,;\,\rho]$ where $2 = (2, 2)$. Then $B = A[X\,;\,\rho]/(X^2 - 2)$ is a 2-strongly cyclic extension of A with $C(B) = Q = \{(q, q)\,;\, q \in Q\}$.

3. Connected strongly abelian extensions. Let $G = (\sigma_1) \times (\sigma_2) \times \cdots (\sigma_m)$ be an abelian group of order n such that $|\sigma_i| = n_i$. If G is a p-group and A is of characteristic p, then the connectedness of a G-abelian extension B of A implies that of $B_i = B^{\sigma_i}$ for $i = 1, \ldots, m$ and the connectedness of each B_i implies that of B (see [1]). But the following examples show that the above are not valid when B is a G-strongly abelian extension.

Examples 3.1. (i) Let $A = Q[\sqrt{-1}]$ and $R = A[X_1, X_2 ; \{-1\}] = \{\sum X_1^{\nu_1} X_2^{\nu_2} a_{\nu_1\nu_2} ; a_{\nu_1\nu_2} \in A\}$ such that $aX_i = X_i a$ and $X_2X_1 = -X_1X_2$. Then $M = (X_1^2-1, X_2^2-1)R$ is a two sided ideal of R and $B = A[x_1, x_2 ; \{-1\}] = R/M$ becomes a G-strongly abelian extension witn respect to $G = (\sigma_1)\times(\sigma_2)$ by $\sigma_i(x_j) = (-1)^\varepsilon x_j$ where $\varepsilon = \delta_{ij}$. Then $C(B) = Q[\sqrt{-1}]$ is connected, $B^{\sigma_1} = A \oplus x_2 A = C(B^{\sigma_1})$ and $e = 1/2(1+x_2)$ is a nontrivial idempotent of $C(B^{\sigma_1})$.

(ii) Let $K = Q[\sqrt{-1}]$, $A = K[\sqrt{3}]$, $\rho(k_0+k_1\sqrt{3}) = k_0-k_1\sqrt{3}$ and $R = A[X_1, X_2 ; \rho_1 = \rho_2 = \rho]$. Then $M = (X_1^2-2, X_2^2-2)R$ is a two sided ideal of R and $B = A[x_1, x_2 ; \rho_1 = \rho_2 = \rho] = R/M$ becomes a G-strongly abelian extension with respect to $G = (\sigma_1)\times(\sigma_2)$ by $\sigma_i(x_j) = (-1)^\varepsilon x_j$. Then $C(B) = K \oplus x_1x_2K$ and $e = 1/2(1+1/2(x_1x_2))$ is a nontrivial idempotent of $C(B)$. On the other hand, $C(B^{\sigma_1}) = C(B^{\sigma_2}) = K$ is connected.

Hereafter we put B is a G-strongly abelian extension of a connected ring A such that $B = A[x_1, \dots, x_m ; \rho_1, \dots, \rho_m]$ and $x_i^{n_i} = \alpha_i$ (i.e., $ax_i = x_i\rho_i(a)$ for $a \in A$ and $x_ix_j = x_jx_i$).

Let r_i be the product of distinct prime factors of n_i. Then B is connected if $T = B^H$ is connected for $H = (\sigma_1^{r_1})\times\cdots\times(\sigma_m^{r_m})$. Therefore we may assume that $n_i = \prod_{j=1}^{s} p_j^{e_{ji}}$, $e_{ji} = 1$ or 0 for all $i = 1, \dots, m$ to study the connectedness of B. We now put

$$A_k = A[x_1, \dots, x_k ; \rho_1, \dots, \rho_k]$$
$$Z_k = \bigcap_{j=1}^{k} U(A)^{\rho_j}.$$

Then we have the following

Lemma 3.2. *Let A_{k-1} be connected and $U(A)^{\rho_k^q} = U(A)^{\rho_k}$ for some $q = n_k/p$ where p is a prime factor of n_k. Then the following conditions are equivalent.*

(1) *$Y^p - \alpha_k$ is w-irreducible in $A_{k-1}[Y ; \rho_k{}^{*q}]$.*

(2) *$\alpha_k \notin \Lambda_p = \{\alpha_1^{\mu_1}\cdots\alpha_{k-1}^{\mu_{k-1}}a^p ; \mu_j$ is an integer such that $n_j\mu_j = p\nu_j$ for some integer ν_j, $a \in Z_k$ and $\tilde{a}\rho_k^q = \rho_{k-1}^{\nu_{k-1}}\cdots\rho_1^{\nu_1}\}$.*

Proof. (1) → (2). If $\alpha_k \in \Lambda_p$, then $\alpha_k = \alpha_1^{\mu_1}\cdots\alpha_{k-1}^{\mu_{k-1}}a^p$ where each μ_j is an integer such that $n_j\mu_j = p\nu_j$ for some integer ν_j. Hence we have $\alpha_k = (x_1^{\nu_1}\cdots x_{k-1}^{\nu_{k-1}}a)^p$, and $Y^p - \alpha_k = (Y-\beta)(Y^{p-1}+Y^{p-2}\beta+\cdots+\beta^{p-1})$ where $\beta = x_1^{\nu_1}\cdots x_{k-1}^{\nu_{k-1}}a$. Since $Y-\beta$ and $Y^{p-1}+y^{p-2}\beta+\cdots+\beta^{p-1}$ are generators of $A_{k-1}[Y ; \rho_k{}^{*q}]$ by the conditions $a \in Z_k$ and $\tilde{a}\rho_k^q = \rho_{k-1}^{\nu_{k-1}}\cdots\rho_1^{\nu_1}$, $Y^p - \alpha_k$ is not w-irreducible.

(2) $\rightarrow$ (1). Assume $Y^p - \alpha_k$ is not w-irreducible. Then there exists $f \in U(A_{k-1})$ such that $Y - f$ is a generator and a factor of $Y^p - \alpha_k$ by Corollary 2.3. Hence f satisfies

(i) $\rho_j^*(f) = f$ for $1 \leqq j \leqq k-1$

(ii) $\rho_k^{*q}(f) = f$

(iii) $gf = f\rho_k^{*q}(g)$ for each $g \in A_{k-1}$.

Noting that $\rho_j^* \sigma_i = \sigma_i \rho_j^*$ for each i, j, we can see that $f^{-1}\sigma_j(f) = \sigma_j(f)f^{-1}$ and $f^{-1}\sigma_j(f) \in C(A_{k-1})$ for $j = 1,\ldots,m$ by (i) $-$ (iii). Further $(f^{-1}\sigma_j(f))^p = (f^{-p}\sigma_j(f^p)) = \alpha_k^{-1}\alpha_k = 1$ show that $\sigma_j(f) = f\eta_j$ for $\eta_j \in \{\zeta^i ; i = 1,\ldots, n\}$ by ([2, Corollary 2.5]). Thus

$$f = x_1^{\nu_1}\cdots x_{k-1}^{\nu_{k-1}}a \text{ for } a \in Z_k \text{ and } \tilde{a}\rho_k^{\,q} = \rho_{k-1}^{\nu_{k-1}}\cdots\rho_1^{\,\nu_1}.$$

Consequently, we have

$\alpha_k = f^p = (x_1)^{p\nu_1}\cdots(x_{k-1})^{p\nu_{k-1}}a^p$. Since $\{x_1^{\xi_1}\cdots x_{k-1}^{\xi_{k-1}} ; 0 \leqq \xi_i \leqq n_i - 1\}$ is linearly independent over A, this means that $p\nu_j = n_j\mu_j$ for some μ_j and $\alpha_k = \alpha_1^{\mu_1}\cdots\alpha_{k-1}^{\mu_{k-1}}a^p$.

Corollary 3.3. *Let A be connected. If there exist automorphisms $\rho_1,\ldots,\rho_m$ of A and elements $\alpha_1,\ldots,\alpha_m \in U(A)$ such that*

(i) $\rho_i^{n_i} = \tilde{\alpha}_i^{-1}$, $\rho_j(\alpha_i) = \alpha_i$, $\rho_j(\zeta) = \zeta$ *for* $i, j = 1,\ldots,m$,

(ii) $U(A)^{\rho_i^{\,q}} = U(A)^{\rho_i}$ $(i = 1,\ldots,m)$ *for each prime factor p and* $q = n_i/p$,

(iii) *For each prime factor p of n_k $(k = 1,\ldots,m)$, $\alpha_k \notin \Lambda_p = \{\alpha_1^{\mu_1}\cdots\alpha_{k-1}^{\mu_{k-1}}a^p$; each μ_j is an integer such that $n_j\mu_j = p\nu_j$ for some integer ν_j, $a \in Z_k$ and $\tilde{a}\rho_k^{\,q} = \rho_{k-1}^{\nu_{k-1}}\cdots\rho_1^{\,\nu_1}$ for $q = n_k/p\}$,*

then A has a connected G-strongly abelian extension B.

Proof. By (i), $B = A[X_1,\ldots,X_m ; \rho_1,\ldots,\rho_m]/(X_1^{n_1} - \alpha_1,\ldots,X_m^{n_m} - \alpha_m)$ is a G-strongly abelian extension of A. Then (ii) and (iii) show that each A_k is connected by Lemma 3.2 and Theorem 2.5.

Let A be a commutative ring. Then it is known that a commutative G-abelian extension of A is a G-strongly abelian extension. Hence, if B is a commutative G-abelian extension of A, then B is obtained by $A[X_1,\ldots,X_m]/(X_1^{n_1} - \alpha_1,\ldots,X_m^{n_m} - \alpha_m)$ for $\alpha_i \in U(A)$, $i = 1,\ldots,m$. Assume now A_{k-1} is connected and $Y^p - \alpha_k$ is not w-irreducible in $A_{k-1}[Y]$ for some prime factor p of n_k. Then, as is shown in the proof of Lemma 3.2.(2), there exists

$f = x_1^{\nu_1}\cdots x_{k-1}^{\nu_{k-1}}a$, $a \in U(A)$ such that $\alpha_k = f^p = x_1^{p\nu_1}\cdots x_{k-1}^{p\nu_{k-1}}a^p$, and hence $p\nu_j = n_j\mu_j$ for some μ_j. Let $\mu_j = ph_j + s_j$ $(0 \leqq s_j < p)$. Then $x_j^{p\nu_j} = x_j^{n_j\mu_j} = \alpha_j^{s_j}\beta_j$ where $\beta_j = (\alpha_j^{h_j})^p$. Hence we may put $\Lambda_p = \{\alpha_1^{\mu_1}\cdots\alpha_{k-1}^{\mu_{k-1}}U(A)^p;\ 0 \leqq \mu_i < p\}$ in Lemma 3.2. Combining this with Lemma 3.2, we have the following

Theorem 3.4. *Let A be a connected commutative ring.*

(a) *A has a connected commutative G-abelian extension B if and only if there exist $\alpha_1,\dots,\alpha_m \in U(A)$ such that $\alpha_i U(A)^p$ is a distinct cosets from $\{\alpha_1^{\mu_1}\cdots\alpha_{i-1}^{\mu_{i-1}}\alpha_{i+1}^{\mu_{i+1}}\cdots\alpha_m^{\mu_m}U(A)^p;\ 0 \leqq \mu_i \leqq n_i-1\}$ in $U(A)/U(A)^p$ for each prime factor p of n_i and $i = 1,2,\dots,m$.*

(b) *If each $n_i = p_1p_2\cdots p_s$, then A has a connected commutative G-abelian extension B if and only if there exist elements $\alpha_1,\dots,\alpha_m \in U(A)$ such that $\alpha_1^{\mu_1}\cdots\alpha_m^{\mu_m}U(A)^{p_i}$ $(0 \leqq \mu_j < p_j)$ are distinct cosets in $U(A)/U(A)^{p_i}$ for $i = 1,\dots,s$.*

Proof. (a) is a direct consequence of Lemma 3.2 and Corllary 3.3.

(b) Let $B \cong A[X_1,\dots,X_m]/(X_1^{n_1}-\alpha_1,\dots,X_m^{n_m}-\alpha_m)$ be a connected commutative G-abelian extension of A. If $\alpha_1^{\mu_1}\cdots\alpha_m^{\mu_m}U(A)^{p_i} = \alpha_1^{\nu_1}\cdots\alpha_m^{\nu_m}U(A)^{p_i}$ for some j with $\mu_j \neq \nu_j$, then we may assume $\alpha_j = \alpha_1^{\xi_1}\cdots\alpha_{j-1}^{\xi_{j-1}}\alpha_{j+1}^{\xi_{j+1}}\cdots\alpha_m^{\xi_m}c^{p_i}$ for some $c \in U(A)$ and $0 \leqq \xi_i < p_i$ since $\alpha_j^{\mu_j-\nu_j}U(A)^{p_i}$ is a generator of a cyclic group $(\alpha_jU(A)^{p_i})$ in $U(A)/U(A)^{p_i}$. But this means that $\alpha_j = ((x_1^{n_1/p_i})^{\xi_1}\cdots(x_{j-1}^{n_{j-1}/p_i})^{\xi_{j-1}}(x_{j+1}^{n_{j+1}/p_i})^{\xi_{j+1}}\cdots(x_m^{n_m/p_i})^{\xi_m}c)^{p_i}$ and this contradicts to the irreduciblity of $X_j^{p_j}-\alpha_j$ in $A[x_1,\dots,x_{j-1},x_{j+1},\dots x_m][X_j]$. Conversely, if there exist $\alpha_1,\dots,\alpha_m \in U(A)$ which satisfy the condition, then $B = A[X_1,\dots,X_m]/(X_1^{n_1}-\alpha_1,\dots,X_m^{n_m}-\alpha_m)$ is a connected commutative G-abelian extension of A.

When A is a two sided simple ring, we can characterize a connected G-strongly abelian extension $S = A[x_1,\dots,x_m;\ \rho_1,\dots,\rho_m,\mathscr{A}]$ as follow.

Theorem 3.5. *Let A be a two sided simple ring (resp. a simple artinian ring). Then a G-strongly abelian extension S of A is a two sided simple ring (resp. a simple artinian ring) if and only if S is connected.*

Proof. Since $A[X_i;\ \rho_i]/(X_i^{n_i}-\alpha_i)$ is a finite direct sum of two sided simple rings (resp. simple artinian rings) by Theorem 2.8, we can see that S is also a finite direct sum of two sided simple rings (resp. simple artinian rings) by inductive argument. Thus S is a two sided simple ring (resp. a simple artinian ring) if and only if S is connected.

References

[1] M. Ferrero and K. Kishimoto : On connectedness of p-Galois extensions of rings, Math. J. Okayama Univ. **25** (1983), 103–121.

[2] G. J. Janusz : Separable algebras over commutative rings, Trans. Amer. Math. Soc., **122** (1966), 461–479.

[3] K. Kishimoto : On abelian extensions of rings II, Math. J. Okayama Univ. **15** (1971), 57–70.

[4] K. Kishimoto : On p-extensions of an algebra of characteristic p, J. Algebra, **88** (1984), 173–177.

[5] T. Nagahara and A. Nakajima : On strongly cyclic extensions of commutative rings, Math. J. Okayama Univ. **15** (1971), 91–100.

[6] Y. Miyashita : On finite outer Galois theory of rings, J. Fac. Sci. Hokkaido Univ. **19** (1966), 114–134.

[7] Y. Miyashita : On a skew polynomial ring, J. Math. Soc. Japan, **31** (1979), 317–330.

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