Math. J. Okayama Univ. 26 (1984), 49–57

AZUMAYA ALGEBRAS AND SKEW POLYNOMIAL RINGS. II

Dedicated to Professor Yoshio Matsuoka on his 60th birthday

SHŪICHI IKEHATA

In [5], we have studied some Azumaya algebras induced by skew polynomial rings over commutative rings. In this paper, we shall continue our study on some skew polynomial rings of automorphism type whose coefficient rings are Azumaya algebras. The main result of this paper is the following: Let B be an Azumaya C-algebra, and ρ an automorphism of B. Assume that the order of the cyclic group G generated by $\rho \mid C$ (the restriction of ρ to C) is $m \geqq 2$, and C/C^G is a G-Galois extension. If there exists an invertible element u in B such that $au = u\rho^m(a)$ $(a \in B)$ and $\rho(u) = u$, then the skew Laurent polynomial ring $B[X, X^{-1}; \rho]$ is an Azumaya $C^G[X^m u^{-1}, X^{-m}u]$-algebra (See Theorem 2). As a corollary of this theorem, we have the following: Let B be a commutative ring, ρ an automorphism of B of order $m \geqq 2$. Let G be the cyclic group generated by ρ, and $A = B^G$. Then B/A is a G-Galois extension if and only if $B[X, X^{-1}; \rho]$ is an Azumaya $A[X^m, X^{-m}]$-algebra.

As applications of Theorem 2, we shall prove a generalization of R. Irving's theorem which is one of the main results in [6], and some results concerning skew polynomial rings in several variables which generalize Amitsur and Saltman [1, Theorem 1.3] and Szeto [10, Theorem 3.2].

Throughout this paper, B will mean a ring, ρ an automorphism of B. We denote by $B[X; \rho]$ (resp. $B[X, X^{-1}; \rho]$) the skew polynomial ring (resp. skew Laurent polynomial ring) defined by $aX = X\rho(a)$ $(a \in B)$. By $B[X; \rho]_{(0)}$, we denote the set of all monic polynomials g in $B[X; \rho]$ with $gB[X; \rho] = B[X; \rho]g$. A ring extension B/A is called separable if the B-B-homomorphism of $B \otimes_A B$ onto B defined by $a \otimes b \to ab$ splits, and B/A is called *H-separable* if $B \otimes_A B$ is B-B-isomorphic to a direct summand of a finite direct sum of copies of B. As is well known, every H-separable extension is separable. A polynomial g in $B[X; \rho]_{(0)}$ is called *separable* (resp. *H-separable*) if $B[X; \rho]/gB[X; \rho]$ is a separable (resp. H-separable) extension of B. Furthermore, a ring extension B/A is called *G-Galois* if there exists a finite group G of automorphism of B such that $A = B^G$ (the fixed ring of G in B) and $\sum_i x_i\sigma(y_i) = \delta_{1,\sigma}$ ($\sigma \in B$) for some finite number of elements x_i, y_i in B. We shall use the following conventions : $U(B) =$ the set of all invertible elements in the ring B ; u_l (resp. u_r) $=$ the left (resp. right) multiplication effected by $u \in B$; $B^\rho = \{a \in B \mid \rho(a) = a\}$.

First, we state the following lemma which is useful in the proof of Theorem 2.

Lemma 1. *Let $f = X^m + X^{m-1}a_{m-1} + \cdots + Xa_1 + a_0$ be in $B[X;\rho]_{(0)}$, and $m \geq 2$. If f is H-separable in $B[X;\rho]$ then there holds the following :*

(1) *Let u be an element in B with $\rho(u) = u$, and $1 \leq j \leq m-1$. If $au = u\rho^j(a)$ (or $\rho^j(a)u = ua$) for all $a \in B$ then $u = 0$.*

(2) *$f = X^m + a_0$, a_0 is invertible in B, and $\rho^m = (a_0^{-1})_l(a_0)_r$.*

(3) *Every H-separable polynomial of degree ≥ 2 in $B[X;\rho]$ is of the form $X^m + a_0c$, where c is an invertible element in the center of B with $\rho(c) = c$.*

Proof. (1) By [5, Theorem 1.1], there exist $y_i, z_i \in B[X;\rho]$ with $\deg y_i < m$ and $\deg z_i < m$ such that $ay_i = y_ia$, $\rho^{m-1}(a)z_i = z_ia$ ($a \in B$) and $X^{m-1}\sum_i \rho^{*m-1}(y_i)z_i = \sum_i \rho^{*m-1}(y_i)z_iX^{m-1} \equiv 1$, $X^k\sum_i \rho^{*k}(y_i)z_i \equiv 0$ (mod $fB[X;\rho]$) $(0 \leq k \leq m-2)$, where $\rho^* : B[X;\rho] \to B[X;\rho]$ be the automorphism defined by $\rho^*(\sum_i X^ib_i) = \sum_i X^i\rho(b_i)$. Since $u\rho^j(a) = au$ ($a \in B$) and $\rho(u) = u$, we have $u\rho^{*j}(y) = yu$ ($y \in B[X;\rho]$). Then

$$\begin{aligned} u &= X^{m-1}\textstyle\sum_i u\rho^{*m-1}(y_i)z_i = X^{m-1}\sum_i \rho^{*m-j-1}(y_i)uz_i \\ &= X^{m-1}\textstyle\sum_i \rho^{*m-j-1}(y_i)z_iu \equiv 0. \end{aligned}$$

(2) By [4, Lemma 1.3] and [5, Proposition 1.2], we have $aa_i = a_i\rho^{m-i}(a)$ and $\rho(a_i) = a_i$ $(0 \leq i \leq m-1)$. Hence, by (1), $a_i = 0$ $(1 \leq i \leq m-1)$. The rest has been obtained in [5, Proposition 1.2].

(3) follows from (1) and (2).

Now, we shall prove the following theorem which is one of our main results in this paper.

Theorem 2. *Let B be an Azumaya C-algebra, and ρ an automorphism of B. Assume that the order of the cyclic group G generated by $\rho \mid C$ is $m \geq 2$, and C/C^G is a G-Galois extension. If there exists an invertible element u in B such that $au = u\rho^m(a)$ ($a \in B$) and $\rho(u) = u$, then there holds the following :*

(1) *$B[X, X^{-1};\rho]$ is an Azumaya $C^G[t, t^{-1}]$-algebra, where $t = X^mu^{-1}$.*

(2) *$B[X;\rho]_{(0)} = \{X^s g(X^m u^{-1}) u^{\deg g} \mid g(Y)$ is a monic polynomial in $C^G[Y]$ and $s \geq 0\}$.*

(3) *The set of all H-separable polynomials of degree ≥ 2 in $B[X;\rho]$ coincides with $\{X^m - v \mid v \in uU(C^G)\}$. For any $v \in uU(C^G)$, $B[X;\rho]/(X^m - v)B[X;\rho]$ is an Azumaya C^G-algebra.*

(4) *For any $f \in B[X;\rho]_{(0)}$, f is a separable polynomial in $B[X;\rho]$ if and only if $B[X;\rho]/fB[X;\rho]$ is a separable C^G-algebra.*

(5) *The set of all separable polynomials in $B[X;\rho]$ coincides with $\{X^s g(X^m u^{-1}) u^{\deg g} \mid g(Y)$ is a separable polynomial in $C^G[Y]$ with constant term invertible, and $s = 0$ or $1\}$. Let $f = g(X^m u^{-1}) u^{\deg g}$ be a separable polynomial in $B[X;\rho]$, and $z = X^m u^{-1} + fB[X;\rho] \in B[X;\rho]/fB[X;\rho]$. Then $B[X;\rho]/fB[X;\rho]$ is a separable C^G-algebra and its center $C^G[z]$ is isomorphic to a separable C^G-algebra $C^G[Y]/g(Y)C^G[Y]$.*

Proof. (1) As is easily verified, $t = X^m u^{-1}$ is contained in the center of $B[X;\rho]$. Then the central localization $B[X;\rho]_t$ coincides with $B[X, X^{-1};\rho]$, since $X^{-1} = (X^{m-1}u^{-1}) \cdot t^{-1}$. Let $\bar{\rho} : B[t, t^{-1}] \to B[t, t^{-1}]$ be the automorphism defined by $\bar{\rho}(\sum_i t^i b_i) = \sum_i t^i \rho(b_i)$, and set $h = W^m - tu = W^m - X^m \in B[t, t^{-1}][W;\rho] = R'$, where W is an indeterminate and $\beta W = W\bar{\rho}(\beta)$ $(\beta \in B[t, t^{-1}])$. Then h is in $R'_{(0)}$, and $B[X, X^{-1};\rho] = B[X;\rho]_t$ is $B[t, t^{-1}]$-ring isomorphic to R'/hR'. Moreover, the center of $B[t, t^{-1}]$ is $C[t, t^{-1}]$ and $C[t, t^{-1}]/C^G[t, t^{-1}]$ is a $\langle \bar{\rho} \mid C[t, t^{-1}] \rangle$-Galois extension. Since the order of $\langle \bar{\rho} \mid C[t, t^{-1}] \rangle$ is m, [5, Proposition 1.4] shows that h is an H-separable polynomial in R'. We set here $S = R'/hR'$. Then $S \supseteq B[t, t^{-1}] \supseteq C[t, t^{-1}] \supseteq C^G[t, t^{-1}]$, $S/B[t, t^{-1}]$ is an H-separable extension, $B[t, t^{-1}]$ is an Azumaya $C[t, t^{-1}]$-algebra, and $C[t, t^{-1}]/C^G[t, t^{-1}]$ is a G-Galois extension. Hence, by the transitivity of separability, we see that S is a separable $C^G[t, t^{-1}]$-algebra. We shall now show that the center of S is $C^G[t, t^{-1}]$. Obviously, $C^G[t, t^{-1}]$ is contained in the center of S. Conversely, let $d = \sum_{i=0}^{m-1} w^i d_i$ $(d_i \in B[t, t^{-1}])$ be an arbitrary element in the center of S, where $w = W + hR' \in S$. Then $wd = dw$ implies $\bar{\rho}(d_i) = d_i$, and $\beta d = d\beta$ $(\beta \in B[t, t^{-1}])$ does $\bar{\rho}^i(\beta) d_i = d_i(\beta)$ $(0 \leq i \leq m-1)$. Since h is an H-separable polynomial in R', we get $d_i = 0$ $(1 \leq i \leq m-1)$, by Lemma 1 (1). Thus, the center of S is $C^G[t, t^{-1}]$, and hence $B[X, X^{-1};\rho]$ is an Azumaya $C^G[t, t^{-1}]$-algebra.

(2) Let $f \in B[X;\rho]_{(0)}$. We may write $f = X^s(X^n + X^{n-1}a_{n-1} + \cdots + Xa_1 + a_0)$, $a_0 \neq 0$ and $n = qm + r$ $(0 \leq r < m)$. Then, noting that f, $X^s \in B[X;\rho]_{(0)}$, we can easily see that $X^n + X^{n-1}a_{n-1} + \cdots + Xa_1 + a_0 \in B[X;$

$\rho]_{(0)}$. Hence, by [4, Lemma 1.3], $aa_0 = a_0\rho^n(a) = a_0u^{-q}\rho^r(a)u^q$ and $aa_i = a_i\rho^{n-i}(a)$ $(a \in B)$. In particular, $(\rho^r(c)-c)a_0u^{-q} = 0$ for any $c \in C$. Let I be the ideal in C generated by $\{\rho^r(c)-c \mid c \in C\}$. We shall prove $r = 0$. Actually, if not, since C/C^G is G-Galois, we have $ann_C I = 0$. On the other hand, since $ann_B IB \neq 0$, we have $ann_C I = C \cap ann_B IB \neq 0$ by [2, p.54, Corollary 3.7], which is a contradiction. Thus, we have seen that $r = 0$, that is, $n = qm$. Similarly, we can show that $a_i = 0$ for every i which is not a multiple of m. Moreover, since $a_{n-1} = 0$, we have $\rho(a_i) = a_i$ $(0 \leq i \leq n-1)$ by [4, Remark 1.4]. Hence $a_{jm} = c_ju^{q-j}$ with some $c_j \in C^G$ $(0 \leq j \leq q-1)$. Putting $g(Y) = Y^q + Y^{q-1}c_{q-1} + \cdots + Yc_1 + c_0 \in C^G[Y]$, we have $X^n + X^{n-1}a_{n-1} + \cdots + a_0 = g(X^mu^{-1})u^q$.

(3) The first assertion follows from [5, Proposition 1.4]. Let $v \in uU(C^G)$. Then the canonical map $B[X;\rho] \to B[X;\rho]/(X^m-v)B[X;\rho]$ can be extended to a B-ring epimorphism $B[X, X^{-1};\rho] \to B[X;\rho]/(X^m-v)B[X;\rho]$. By (1), $B[X,X^{-1};\rho]$ is an Azumaya algebra over its center $C^G[X^mu^{-1}, X^{-m}u] = C^G[X^mv^{-1}, X^{-m}v]$. Hence, by [2, p.46, Proposition 1.11], $B[X;\rho]/(X^m-v)B[X;\rho]$ is an Azumaya C^G-algebra.

(4) is clear by the transitivity of separability.

(5) Let $g(Y)$ be a monic polynomial of degree q in $C^G[Y]$ with constant term invertible. Then, by (2), $g(X^mu^{-1})u^q$ is in $B[X;\rho]_{(0)}$. We set $S' = B[X;\rho]/g(X^mu^{-1})u^qB[X;\rho]$. Since the constant term of g is invertible, the canonical homomorphism $B[X;\rho] \to S'$ can be extended to a B-ring epimorphism $B[X, X^{-1};\rho] \to S'$. Hence, by (1) and [2, p.46, Proposition 1.11], we see that S' is an Azumaya $C^G[z]$-algebra, where $z = X^mu^{-1} + g(X^mu^{-1})u^qB[X;\rho] \in S'$. Since $C^G[z]$ is C^G-isomorphic to $C^G[Y]/g(Y)C^G[Y]$, S' is a separable C^G-algebra if and only if $g(Y)$ is a separable polynomial in $C^G[Y]$, by [2, p.55, Theorem 3.8].

Let $g(X^mu^{-1})u^q$ be separable in $B[X;\rho]$. Then noting that $B[X;\rho]/Xg(X^mu^{-1})u^qB[X;\rho] \simeq B \oplus (B[X;\rho]/g(X^mu^{-1})u^qB[X;\rho])$, we see that $Xg(X^mu^{-1})u^q$ is also separable.

Now, let f be a separable polynomial in $B[X;\rho]$. Then, by (2), $f = X^sg_0(X^mu^{-1})u^{q_0}$ with some monic polynomial $g_0(Y)$ of degree q_0 in $C^G[Y]$. Then, by [3, Lemma 1], $s = 0$ or 1 and the constant term of g_0 is invertible. If $f = Xg_0(X^mu^{-1})u^{q_0}$ then $g_0(X^mu^{-1})u^{q_0}$ is a separable polynomial in $B[X;\rho]$ by [8, Theorem 1.11]. Hence, by the above statement, we see that $g_0(Y)$ is a separable polynomial in $C^G[Y]$.

As a special case of Theorem 2, we have the following which includes

[5, Lemmas 2.5, 2.6 and Theorem 2.7].

Corollary 3. *Let B be a commutative ring, and ρ an automorphism of order $m \geq 2$. Let G be the cyclic group generated by ρ, and $A = B^G$. Then B/A is a G-Galois extension if and only if $B[X, X^{-1}; \rho]$ is an Azumaya $A[X^m, X^{-m}]$-algebra. When this is the case, there holds the following:*

(1) *$B[X; \rho]_{(0)} = \{X^s g(X^m) \mid g(Y)$ is a monic polynomial in $A[Y]$ and $s \geq 0\}$.*

(2) *The set of all H-separable polynomials of degree ≥ 2 in $B[X; \rho]$ coincides with $\{X^m - v \mid v \in U(A)\}$. For any $v \in U(A)$. $B[X; \rho]/(X^m - v)B[X; \rho]$ is an Azumaya A-algebra.*

(3) *For any $f \in B[X; \rho]_{(0)}$, f is separable in $B[X; \rho]$ if and only if $B[X; \rho]/fB[X; \rho]$ is a separable A-algebra.*

(4) *The set of all separable polynomials in $B[X; \rho]$ coincides with $\{X^s g(X^m) \mid g(Y)$ is a separable polynomial in $A[Y]$ with constant term invertible, and $s = 0$ or $1\}$. Let $f = g(X^m)$ be a separable polynomial in $B[X; \rho]$, and $z = X^m + fB[X; \rho] \in B[X; \rho]/fB[X; \rho]$. Then $B[X; \rho]/fB[X; \rho]$ is a separable A-algebra and its center $A[z]$ is isomorphic to a separable A-algebra $A[Y]/g(Y)A[Y]$. Moreover, $B[z]$ is a maximal commutative $A[z]$-algebra such that*

$$B[X; \rho]/fB[X; \rho] \otimes_{A[z]} B[z] \simeq M_m(B[z]).$$

Proof. Assume that $B[X, X^{-1}; \rho]$ is an Azumaya $A[X^m, X^{-m}]$-algebra. Let $\bar{\rho}: B[X^m, X^{-m}] \to B[X^m, X^{-m}]$ be the automorphism defined by

$$\bar{\rho}(\textstyle\sum_i (X^m)^i b_i) = \sum_i (X^m)^i \rho(b_i),$$

and

$$h = W^m - X^m \in B[X^m, X^{-m}][W; \rho] = R',$$

where W is an indeterminate. Then $B[X, X^{-1}; \rho]$ is $B[X^m, X^{-m}]$-ring isomorphic to R'/hR'. Hence $B[X^m, X^{-m}]/A[X^m, X^{-m}]$ is a $\langle \bar{\rho} \rangle$-Galois extension. Then, considering the B-ring homomorphism $B[X^m, X^{-m}] \to B$ defined by $X^m \to 1$, we see that B/A is a G-Galois extension. The rest of the proof is clear by Theorem 2 and [5, Theorem 2.2].

As an application of Theorem 2, we shall prove a theorem which generalizes a theorem of Irving [6, Theorem 5.9]. Let C be a commutative ring, and ρ an automorphism of C. An ideal I of C is called ρ-invariant if $\rho(I)$

$\subseteq I$. A ρ-invariant ideal I is called ρ-prime provided for any ρ-invariant ideals J and K, $JK \subseteq I$ implies $J \subseteq I$ or $K \subseteq I$; C is a ρ-prime ring if (0) is a ρ-prime ideal. If C is ρ-prime then every nonzero element of $Q = C^{\rho}$ is not a zero-divisor in C; in particular, Q is an integral domain. The proof of [6, Theorem 5.9] together with [9, Lemma 1.1] gives the following

Lemma 4. *Let C be a ρ-prime ring, and F the quotient field of $Q = C^{\rho}$. Let $G = \langle \rho \otimes 1_F \rangle$, and $Q^* = Q - \{0\}$. If ρ is of order m, then G is of order m and $C_{Q^*} = C \otimes_Q F$ is a G-Galois extension over F.*

We are now ready to prove the following generalization of [6, Theorem 5.9].

Theorem 5. *Let B be an Azumaya C-algebra, and ρ an automorphism of B. Assume that $\rho | C$ is of order m and C is $\rho | C$-prime. Let F denote the quotient field of $Q = C^{\rho | C}$. If there exists an invertible element u in B such that $au = u\rho^m(a)$ $(a \in B)$ and $\rho(u) = u$. Then $(B \otimes_Q F)[X, X^{-1}; \rho \otimes 1_F]$ is an Azumaya $F[X^m u^{-1}, X^{-m} u]$-algebra.*

Proof. Obviously, $B \otimes_Q F$ is an Azumaya $C \otimes_Q F$-algebra with an automorphism $\rho \otimes 1_F$. Then, by Lemma 4, $G = \langle (\rho | C) \otimes 1_F \rangle$ is of order m and $C \otimes_Q F$ is a G-Galois extension of F. Hence the assertion follows from Theorem 2.

Corollary 6 ([6, Theorem 5.9]). *Let C be a commutative ring, and ρ an automorphism of C. Assume that ρ is of order m and C is a ρ-prime ring. Let F denote the quotient field of $Q = C^{\rho}$. Then $(C \otimes_Q F)[X, X^{-1}; \rho \otimes 1_F]$ is an Azumaya $F[X^m, X^{-m}]$-algebra.*

As another application of Theorem 2, we shall prove some results concerning skew polynomial rings in several variables. Let ρ_i $(1 \leq i \leq e)$ be automorphisms of a ring B, and let u_{ij} $(1 \leq i, j \leq e)$ be invertible elements in B such that

(i) $u_{ij} = u_{ji}^{-1}$ and $u_{ii} = 1$,
(ii) $\rho_i \rho_j \rho_i^{-1} \rho_j^{-1} = (u_{ij})_l (u_{ij}^{-1})_r$,
(iii) $u_{ij} \rho_j(u_{ik}) u_{jk} = \rho_i(u_{jk}) u_{ik} \rho_k(u_{ij})$.

Then the set of polynomials in e indeterminates $\mathbf{B} = B[X_1, \ldots, X_e; \rho_1, \ldots, \rho_e; \{u_{ij}\}] = \{\sum X_1^{\nu_1} X_2^{\nu_2} \cdots X_e^{\nu_e} b_{\nu_1 \nu_2 \cdots \nu_e} | b_{\nu_1 \nu_2 \cdots \nu_e} \in B, \nu_k \geq 0\}$ forms an associative ring if we define the multiplication by the distributive law and the rules $aX_i = X_i\rho_i(a)$ $(a \in B)$ and $X_iX_j = X_jX_iu_{ij}$ (see e.g., [7] or [1]).

Now, assume further that there exist invertible elements u_i $(1 \leq i \leq e)$ in B such that

$$\text{(iv)} \quad \rho_i^{m_i} = (u_i^{-1})_l(u_i)_r$$

and

$$\text{(v)} \quad u_{ji}\rho_i(u_{ji})\cdots\rho_i^{m_i-1}(u_{ji}) = \rho_j(u_i^{-1})u_i \ (1 \leq i \leq e).$$

Then $(X_i^{m_i} - u_i)\mathbf{B}$ is a two-sided ideal in $\mathbf{B}$.
Actually,

$$a(X_i^{m_i} - u_i) = (X_i^{m_i} - u_i)\rho_i^{m_i}(a) \ (a \in B)$$

and

$$X_j(X_i^{m_i} - u_i) = (X_i^{m_i} - u_i)X_ju_{ji}\rho_i(u_{ji})\cdots\rho_i^{m_i-1}(u_{ji}) \ (1 \leq i,j \leq e).$$

The map $\bar{\rho}_i : \mathbf{B} \to \mathbf{B}$ defined by $\bar{\rho}_i(\sum X_1^{\nu_1}\cdots X_e^{\nu_e}b_{\nu_1\cdots\nu_e}) = \sum (X_1u_{1i})^{\nu_1}\cdots(X_eu_{ei})^{\nu_e}\rho_i(b_{\nu_1\cdots\nu_e})$ is an automorphism of $\mathbf{B}$ which is an extension of ρ_i. Then we have

$$\begin{aligned}\bar{\rho}_i(X_j^{m_j}u_j^{-1}) &= (X_ju_{ji})^{m_j}\rho_i(u_j^{-1})\\ &= X_j^{m_j}\rho_j^{m_j-1}(u_{ji})\cdots\rho_j(u_{ji})u_{ji}\rho_i(u_j^{-1})\\ &= X_j^{m_j}u_j^{-1} \quad (\text{by (v)}).\end{aligned}$$

Furthermore, since $au_i = u_i\rho_i^{m_i}(a)$ $(a \in B)$ and $X_ju_i = X_j\rho_j(u_i)u_{ji}\rho_i(u_{ji})\cdots\rho_i^{m_i-1}(u_{ji}) = u_iX_ju_{ji}\rho_i(u_{ji})\cdots\rho_i^{m_i-1}(u_{ji}) = u_i\bar{\rho}_i^{m_i}(X_j)$, $\beta u_i = u_i\bar{\rho}_i^{m_i}(\beta)$ for any $\beta \in \mathbf{B}$, and $X_j^{m_j}u_j^{-1}$ is contained in the center of $\mathbf{B}$.

Now, we shall generalize [1, Theorem 1.3] and [10, Theorem 3.2] as follows:

Theorem 7. *Let B be an Azumaya C-algebra, and let ρ_i $(1 \leq i \leq e)$ be automorphisms of B. Assume that $\rho_i | C$ is of order m_i, $\{\rho_i | C \mid 1 \leq i \leq e\}$ generates an abelian group $G = \langle\rho_1 | C\rangle \times \cdots \times \langle\rho_e | C\rangle$ and that C/C^G is a G-Galois extension. Let u_{ij} $(1 \leq i,j \leq e)$ be invertible elements in B satisfying* (i), (ii) *and* (iii), *and let u_i $(1 \leq i \leq e)$ be invertible elements in B satisfying* (iv) *and* (v). *Then there holds the following*:

(1) *$B[X_1,\ldots,X_e,X_1^{-1},\ldots,X_e^{-1}\,;\,\rho_1,\ldots,\rho_e\,;\,\{u_{ij}\}]$ is an Azumaya $C^G[t_1,\ldots,t_e,t_1^{-1},\ldots,_e^{-1}]$-algebra, where $t_i = X_i^{m_i}u_i^{-1}$.*

(2) *For any invertible elements* $c_i \in C^G$ $(1 \leq i \leq e)$, we put $v_i = c_i u_i$. Then $B[X_1, \ldots, X_e ; \rho_1, \ldots, \rho_e ; \{u_{ij}\}]/(X_1^{m_1} - v_1, \ldots, X_e^{m_e} - v_e)$ $B[X_1, \ldots, X_e ; \rho_1, \ldots, \rho_e ; \{u_{ij}\}]$ *is an Azumaya* C^G*-algebra.*

Proof. (1) We shall prove the theorem by induction on e. When $e = 1$, the assertion is Theorem 2 itself. Let $1 \leq r \leq e$. We put $B_{r-1} = B[X_1, \ldots, X_{r-1}, X_1^{-1}, \ldots, X_{r-1}^{-1} ; \rho_1, \ldots, \rho_{r-1} ; \{u_{ij}\}]$, and $C_{r-1} = C^{G_{r-1}}[t_1, \ldots, t_{r-1}, t_1^{-1}, \ldots, t_{r-1}^{-1}]$, where $G_{r-1} = \langle \rho_1 | C \rangle \times \ldots \times \langle \rho_{r-1} | C \rangle$. Assume that B_{r-1} is an Azumaya C_{r-1}-algebra. As is easily verified, $C^{G_{r-1}}$ is a $\langle \rho_r | C^{G_{r-1}} \rangle$-Galois extension over C^{G_r} and G_r/G_{r-1} is of order m_r. By the fact claimed just before the theorem, we have $\bar{\rho}_r(t_i) = t_i$ $(1 \leq i \leq r-1)$. Hence, C_{r-1} is a $\langle \bar{\rho}_r | C_{r-1} \rangle$-Galois extension over $C^{G_r}[t_1, \ldots, t_{r-1}, t_1^{-1}, \ldots, t_{r-1}^{-1}]$. Further, by (v) and the facts claimed just before the theorem, $\rho_r(u_r) = u_r$ and $u_r = u_r \bar{\rho}_r^{m_r}(\beta)$ for all $\beta \in B_{r-1}$. Then Theorem 2 shows that $B_{r-1}[X_r, X_r^{-1} ; \bar{\rho}_r] = B[X_1, \ldots, X_r, X_1^{-1}, \ldots, X_r^{-1} ; \rho_1, \ldots, \rho_r ; \{u_{ij}\}]$ is an Azumaya algebra over $C_{r-1}^{\rho_r | C_{r-1}}[t_r, t_r^{-1}] = C^{G_r}[t_1, \ldots, t_r, t_1^{-1}, \ldots, t_r^{-1}]$. This completes the induction.

(2) is clear by (1) and [2, p.46, Proposition 1.11].

Corollary 8. *Let B be a commutative ring. Assume that B is an abelian G-Galois extension over A, where $G = \langle \rho_1 \rangle \times \ldots \times \langle \rho_e \rangle$ with ρ_i of order m_i. Let u_{ij} $(1 \leq i, j \leq e)$ be invertible element in B satisfying* (i) *and* (iii), *and let u_i $(1 \leq i \leq e)$ be invertible elements in B satisfying* (v). *Then there holds the following:*

(1) *$B[X_1, \ldots, X_e, X_1^{-1}, \ldots, X_e^{-1} ; \rho_1, \ldots, \rho_e ; \{u_{ij}\}]$ is an Azumaya $A[X_1^{m_1}, \ldots, X_e^{m_e}, X_1^{-m_1}, \ldots, X_e^{-m_e}]$-algebra with a maximal commutative subalgebra $B[X_1^{m_1}, \ldots, X_e^{m_e}, X_1^{-m_1}, \ldots, X_e^{-m_e}]$.*

(2) *For any invertible elements $c_i \in A$, we put $v_i = c_i u_i$ $(1 \leq i \leq e)$. Then $B[X_1, \ldots, X_e ; \rho_1, \ldots, \rho_e ; \{u_{ij}\}]/(X_1^{m_1} - v_1, \ldots, X_e^{m_e} - v_e)$ is an Azumaya A-algebra with a maximal commutative subalgebra B.*

We conclude with the following sharpening of [10, Theorem 3.3].

Theorem 9. *Keep the assumptions and notations of Theorem 7. Assume, in addition, that $\{u_{ij}\} \subseteq C$ and $\{u_i\} \subseteq C$. Then there holds the following:*

(1) *Let $K = C^G[t_1, \ldots, t_e, t_1^{-1}, \ldots, t_e^{-1}]$. Then $B[X_1, \ldots, X_e, X_1^{-1}, \ldots, X_e^{-1} ; \rho_1, \ldots, \rho_e ; \{u_{ij}\}] \simeq B^G[t_1, \ldots, t_e, t_1^{-1}, \ldots, t_e^{-1}] \otimes_K C[X_1, \ldots, X_e, X_1^{-1}, \ldots, X_e^{-1} ; (\rho_1 | C), \ldots, (\rho_e | C) ; \{u_{ij}\}]$ as Azumaya K-algebra, where B^G is the fixed ring of $\langle \rho_1, \ldots, \rho_e \rangle$. Moreover, $B^G[t_1, \ldots, t_e, t_1^{-1}, \ldots, t_e^{-1}]$ is an Azumaya*

K-algebra and B^G is an Azumaya C^G-algebras.

(2) *For any invertible elements $c_i \in C^G$, we put $v_i = c_i u_i$ $(1 \leq i \leq e)$. Then $B[X_1, \dots, X_e ; \rho_1, \dots, \rho_e ; \{u_{ij}\}]/(X_1^{m_1} - v_1, \dots, X_e^{m_e} - v_e)$ $B^G \otimes_{C^G} C[X_1, \dots, X_e ; (\rho_1 | C), \dots, (\rho_e | C) ; \{u_{ij}\}]/(X_1^{m_1} - v_1, \dots, X_e^{m_e} - v_e)$ as Azumaya C^G-algebras.*

Proof. (1) Since C/C^G is a G-Galois extension and B is an Azumaya C-algebra, we see that the centralizer of $C[X_1, \dots, X_e, X_1^{-1}, \dots, X_e^{-1} ; (\rho_1 | C), \dots, (\rho_e | C) ; \{u_{ij}\}]$ in $B[X_1, \dots, X_e, X_1^{-1}, \dots, X_e^{-1} ; \rho_1, \dots, \rho_e ; \{u_{ij}\}]$ is $B^G[t_1, \dots, t_e, t_1^{-1}, \dots, t_e^{-1}]$. By Theorem 7 (1), both $B[X_1, \dots, X_e, X_1^{-1}, \dots, X_e^{-1} ; \rho_1, \dots, \rho_e ; \{u_{ij}\}]$ and $C[X_1, \dots, X_e, X_1^{-1}, \dots, X_e^{-1} ; (\rho_1 | C), \dots, (\rho_e | C) ; \{u_{ij}\}]$ are Azumaya K-algebras. Hence the assertions follow from [2, p.57, Theorem 4.7].

(2) is almost clear.

REFERENCES

[1] S.A. AMITSUR and D. SALTMAN: Generic abelian crossed products and p-algebras, J. Algebra 51 (1978), 76−87.

[2] F. DEMEYER and E. INGRAHAM: Separable Algebras over a Commutative Ring, Lecture Notes in Math. **181**, Springer, Berlin, 1971.

[3] S. IKEHATA: On a theorem of Y. Miyashita, Math. J. Okayama Univ. **21** (1979), 49−52.

[4] S. IKEHATA: On separable polynomials and Frobenius polynomials in skew polynomial rings, Math. J. Okayama Univ. **22** (1980), 115−129.

[5] S. IKEHATA: Azumaya algebras and skew polynomial rings, Math. J. Okayama Univ. **23** (1981), 19−32.

[6] R.S. IRVING: Prime ideals of Ore extensions over commutative rings, J. Algebra **56** (1979), 315−342.

[7] K. KISHIMOTO: On abelian extensions of rings II, Math. J. Okayama Univ. **15** (1971), 57−70.

[8] Y. MIYASHITA: On a skew polynomial ring, J. Math. Soc. Japan **31** (1979), 317−330.

[9] T. NAGAHARA and A. NAKAJIMA: On separable polynomials over a commutative ring IV, Math. J. Okayama Univ. **17** (1974), 49−58.

[10] G. SZETO: On separable abelian extensions of rings, Internat. J. Math. Sci. 5 (1982), 779−784.

DEPARTMENT OF MATHEMATICS
OKAYAMA UNIVERSITY

(*Received August 1, 1984*)