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GALOIS SUBRINGS OF INDEPENDENT AUTOMORPHISM GROUPS OF COMMUTATIVE RINGS ARE QUORITE

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Let R be a commutative ring, G a finite group of automorphisms, let $A = R^G$ be the Galois subring, let G^{ex} denote the canonical extension of G to the quotient ring $Q = Q_{cl}(R)$, and $F = Q^{G^{ex}}$. It is easy to see ([1]) that F is the partial quotient ring of A with respect to the multiplicatively closed subset S of A consisting of all $a \in A$ that are regular in R, that is, $S = A \cap R^*$, and thus that G is *quorite* in the sense that $Q_{cl}(A) = F$ iff R is *torsion free* over A in the sense that $A^* \subseteq R^*$. Sufficient ring-theoretical conditions for this are : (1) R is reduced (= semiprime, or non-singular) ; (2) R is flat over A. As stated, (1) happens if R is semihereditary, and (2) when G is a Galois group.

The purpose of this short note is to report another useful sufficient condition. A group G of automorphisms of R is *independent* provided that the elements of G, considered as functions $R \to R$, are linearly independent over R ([2]).

Theorem. *If G is an independent finite automorphism group of R, then G is quorite.*

Proof. Assume the above notation. Let $a \in A^*$, and I the annihilator ideal of a in R. Let $T_G(x)$ denote the trace of any x in R under G. Since $I \cap A = 0$, evidently $T_G(x) = 0$ for any x in I, and moreover, if r is any element of R, we have then that $xr \in I$, whence $T_G(xr) = 0$, that is,

$$\sum_{g \in G} g(x)g(r) = 0$$

and therefore

$$\sum_{g \in G} g(x)g = 0 \text{ on } R.$$

Since G is independent, it follows that $x = 0$, which proves the theorem.

Example. The converse of the theorem fails. Let R be the direct product of three fields $F_1 \times F_2 \times F_3$, with $F_1 \approx F_2$, and let g denote the extension of this isomorphism to an automorphism of R with Galois subring

$R^g \supseteq F_3$. Since R^g contains an ideal, then by [2] the group (g) is dependent, but quorite since R is reduced.

Question. Let G be a group of automorphisms of a non-commutative ring. If G is finite and independent, is G quorite?

If R is an integral domain, then the answer is yes by [3] without assumption of independence, a result generalized by [4] to any ring R such that R^G is semiprime. See [6] for related results.

The assumption that G is independent is the basis of a number of theorems for what are called strictly Galois extensions in [8], and e.g., [7].

Let $Q = Q^r_{\max}(R)$ denote the maximal right quotient ring of R, and G^{ex} now denote the group of automorphisms of Q which extends G to Q. The theorem of Kitamura [5] states

$$Q^{G^{ex}} = Q^r_{\max}(R)$$

that is, G is *maximally quorite*, if the trace function $R \to R^G$ is *non-degenerate*, that is, does not vanish on any nonzero right ideal.

Corollary. *If R is commutative, and G independent, then G is maximally quorite.*

Proof. As stated in [2], if G is independent, then the trace function is non-degenerate.

In fact, Theorem 8.2 of [2] states that a torsion group G is dependent over R iff for some $g \neq 1$ in G either the fixring R^g contains a nonzero ideal of R, or else the g-trace function is degenerate.

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