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ON A THEOREM OF S. KOSHITANI

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The purpose of this note is to present a theorem which refines the result of S. Koshitani [5, Theorem]. Throughout the present paper, F will represent a field of characteristic $p>0$, and G a finite p-solvable group. Let B be a block ideal of the group algebra FG, and $J(B)$ the Jacobson radical of B. Recently, in [5], S. Koshitani proved that if d is the defect of B, then the least positive integer t such that $J(B)^t=0$ is greater than or equal to $d(p-1)+1$. In view of [3, IV, Lemma 2.2 and Theorem 4.5], [2, Lemma 4.6] and [6, Lemma 12.9], we see that there exists an irreducible B-module a vertex of which is a defect group of B. Hence Koshitani's result follows from the following

Theorem. *Let M be an irreducible FG-module. If a vertex of M has order p^v, then the Loewy length of the projective cover of M is greater than or equal to $v(p-1)+1$.*

All modules considered here are finitely generated right modules. The following notation will be used in the proof of the theorem. Given an FG-module M, we denote by $vx_G(M)$ a vertex of M and by $L(M)$ the Loewy length of M. If H is a subgroup of G, then $M|_H$ is an FH-module obtained from M by restricting the domain of operators to FH. The full matrix ring of degree m over a ring R is denoted by $M_m(R)$. If n is a positive integer, then $\nu(n)$ is the exponent of the highest p-power dividing n.

Proof of Theorem. Let e be a primitive idempotent of FG such that eFG is a projective cover of M. If E is a finite extension field of F, then it is well known that $J(EG)\cong E\otimes_F J(FG)$, and so $E\otimes_F M$ is a completely reducible EG-module. Let $E\otimes_F M = X_1\oplus\cdots\oplus X_r$ be a decompsition of $E\otimes_F M$ into a direct sum of irreducible EG-submodules. Then observing that

$$eEG/eJ(EG)\cong E\otimes_F eFG/eJ(FG)\cong E\otimes_F M,$$

we see that eEG is a projective cover of $E\otimes_F M$, and so eEG is a direct sum of projective covers of X_i $(i=1,\cdots,r)$. It is clear that eEG has the same Loewy length as eFG. Further, by [2, Lemma 4.6], each X_i and M have a vertex in common. Therefore, in order to prove the theorem, we may assume that F contains the cyclotomic field of order $|G|$ over GF(p). Then F

is a splitting field for all subgroups of G. The proof is by induction with respect to $|G|$ and $\nu(|G|)$. Suppose, if possible, G is a minimal counter-example.

Case 1 : Assume that $Q = O_p(G) \neq \langle 1 \rangle$.

Since $Q \subset \mathrm{Ker}\, M$, M becomes an $F\bar{G}$-module, where $\bar{G} = G/Q$, and [4, Lemma 1.3] asserts $vx_{\bar{G}}(M) \underset{\bar{G}}{=} vx_G(M)/Q$. Let $\omega(Q)$ be the augmentation ideal of FQ. Then $FG/\omega(Q)FG \cong F\bar{G}$ and $eFG/e\omega(Q)FG$ is a projective cover of the $F\bar{G}$-module M. Set $v_1 = \nu(|vx_G(M)/Q|)$ and $v_2 = \nu(|Q|)$. Then $L(eFG/e\omega(Q)FG) \geqq v_1(p-1)+1$ by induction. Hence $eJ(FG)^{v_1(p-1)}$ is not contained in $e\omega(Q)FG$. Let $\hat{Q} = \sum_{s \in Q} s$ in FG. Then we have

$$\begin{aligned}\{x \in eFG \mid x\hat{Q} = 0\} &= \{x \in FG \mid x\hat{Q} = 0\} \cap eFG \\ &= FG\omega(Q) \cap eFG \\ &= eFG\omega(Q) = e\omega(Q)FG.\end{aligned}$$

Therefore we get $eJ(FG)^{v_1(p-1)}\hat{Q} \neq 0$. Since $L(FQ)-1 \geqq v_2(p-1)$, we see that $\hat{Q}$ is contained in $\omega(Q)^{v_2(p-1)}$. Hence, noting that $\omega(Q) \subset J(FG)$, we get

$$\begin{aligned}0 \neq eJ(FG)^{v_1(p-1)}\hat{Q} &\subset eJ(FG)^{v_1(p-1)}J(FG)^{v_2(p-1)} \\ &= eJ(FG)^{v(p-1)},\end{aligned}$$

proving that $L(eFG) \geqq v(p-1)+1$. So this case does not occur.

Case 2 : Assume that $O_p(G) = \langle 1 \rangle$.

Let $H = O_{p'}(G)$. Suppose that M belongs to the block ideal B of FG. Let N be an irreducible component of $M|_H$ and let T be the inertial group of N in G:

$$T = \{g \in G \mid N \otimes_{FH} g \cong N \text{ as } FH\text{-modules}\}.$$

At first, suppose that $G \neq T$. Then, by [5, Lemma 1], there exists a block ideal b of FT with block idempotent f and the F-algebra isomorphism ϕ: $B \cong \mathrm{End}(FGf_{FTf})$ given by $[\phi(x)](y) = xy$, $x \in B$, $y \in FGf$. Further, the map sending X to $X^G = X \otimes_{FT} FG$ is a one to one correspondence between irreducible b-modules and irreducible B-modules ([3, V, Theorem 2.5]). We set $t = [G : T]$ and let $\{g_1 = 1, g_2, \cdots, g_t\}$ be a right transversal of T in G. Then $\{f, g_2^{-1}f, \cdots, g_t^{-1}f\}$ is a basis for the free FTf-module FGf. We denote by ψ the isomorphism $\mathrm{End}(FGf_{FTf}) \cong M_t(FTf)$ defined naturally with respect to this basis. Now let X and Y be irreducible b-modules. Then the above together with Frobenius reciprocity theorem implies that

$$\dim_F \mathrm{Hom}_{FT}(Y, X^G|_T) = \dim_F \mathrm{Hom}_{FG}(Y^G, X^G) = \begin{cases} 1 & \text{if } Y \cong X, \\ 0 & \text{if } Y \not\cong X. \end{cases}$$

Hence we see that the socle of $X^G|_T$ is isomorphic to a direct sum of X and irreducible FT-modules which belong to blocks different from b. Therefore, noting that X is isomorphic to a direct summand of $X^G|_T$, we get $X^G f \cong X$. We may assume that X is a minimal right ideal of b, and so we may identify X^G with a right ideal of B generated by X. Then we have

$$[\phi(X^G)](g_i^{-1}f) = (X^G)g_i^{-1}f = (X^G g_i^{-1})f = X^G f = X.$$

for all i, $1 \leqq i \leqq t$. Thus we get

$$\psi\phi(X^G) = \begin{pmatrix} X & X & \cdots & X \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \subset M_t(FTf).$$

Now, we may assume that $M \cong X^G$. Then from the above we get

$$eFG \cong \begin{pmatrix} P & P & \cdots & P \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix},$$

where P is a projective cover of X, and hence we have $L(eFG) = L(P)$. Since $G \neq T$, noting that $vx_G(M) \underset{G}{=} vx_T(X)$, ([1, Theorem 19.16]), we get

$$L(eFG) = L(P) \geqq v(p-1)+1$$

by induction. Next, suppose that $G = T$. Set $\bar{G} = G/H$. Then [7, Theorem 2] asserts that there exists a finite group $\tilde{G}$ and a short exact sequence

$$\langle 1 \rangle \longrightarrow Z \longrightarrow \tilde{G} \longrightarrow \bar{G} \longrightarrow \langle 1 \rangle$$

where Z is a cyclic p'-subgroup in the center of $\tilde{G}$, and there exists a block ideal $\tilde{B}$ of $F\tilde{G}$ such that $B \cong M_n(F) \otimes_F \tilde{B}$ ($n = \dim_F X$). This asserts that there is an irreducible $\tilde{B}$-module $\tilde{M}$ such that $M \cong I \otimes_F \tilde{M}$, where I is an irreducible $M_n(F)$-module. So we have $eFG \cong I \otimes_F \tilde{P}$, where $\tilde{P}$ is a projective cover of $\tilde{M}$, and so we get $L(eFG) = L(\tilde{P})$. Since G is p-solvable and $\nu(|G|) \geqq 1$, it is clear that $O_p(\tilde{G}) \neq \langle 1 \rangle$. Hence, noting that $vx_G(M) \cong vx_{\tilde{G}}(\tilde{M})$, we have

$$L(eFG) = L(\tilde{P}) \geqq v(p-1)+1$$

by Case 1 applied for $\tilde{G}$. So this case does not occur either, and the theorem is proved.

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