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ON THE RADICAL OF AN INFINITE GROUP ALGEBRA

Dedicated to Professor Hirosi Nagao on his 60th birthday

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Throughout the present paper, k will represent an algebraically closed field of characteristic $p>0$, and G a group. In [7], D. A. R. Wallace gave necessary and sufficient conditions for the Jacobson radical $J(kG)$ of the group algebra kG to be commutative under the assumption that p is odd. For the case $p=2$, in [2], we gave conditions for $J(kG)$ to be commutative. However, the problem of which group algebras kG have the radicals of square zero remains unsolved. In § 1, we shall give a condition for $J(kG)$ to be of square zero.

The Jacobson radical $J(Z(kG))$ of the center $Z(kG)$ of kG is a nil ideal ([3, Lemma 4.1.11]), and so it is clear that $J(kG)\supset J(Z(kG))kG$. In case where G is finite, it is well known that the equality $J(kG)=J(Z(kG))\cdot kG$ holds if and only if kG is an Azumaya algebra (over the center), and in [4], Y. Tsushima proved that $J(kG)=J(Z(kG))kG$ holds if and only if the commutator subgroup G' of G is a p'-group. On the other hand, in case where G is an infinite group, it is true that if kG is an Azumaya algebra then $J(kG)=J(Z(kG))kG$, but the converse implication is not necessarily true. Recently, F. R. DeMeyer and G. J. Janusz [1] gave a necessary and sufficient condition for kG to be an Azumaya algebra. In particular, they showed that if kG is an Azumaya algebra, then G' is a finite p'-group. In § 2, we show that if G' has no elements of order a power of p, then $N(kG)=J(Z(kG))\cdot kG$, where $N(kG)$ is the sum of all the nilpotent ideals of kG.

In what follows, we denote by $\Delta(G)$ and $\Delta^p(G)$ the characteristic subgroups of G given by

$$\Delta(G)=\{x\in G\mid [G:C_G(x)] \text{ is finite}\} \text{ and}$$
$$\Delta^p(G)=\langle x\in\Delta(G)\mid x \text{ has order a power of } p\rangle$$

respectively.

1. At first, suppose that $J(kG)\neq 0$ and $J(kG)^2=0$. Then $J(kG)=N(kG)=J(k\Delta^p(G))kG$ ([3, Theorem 8.1.9]) and $\Delta^p(G)$ is finite ([3, Theorem 8.1.12]). Hence it follows from [5, Theorem] that $p=2$ and $|\Delta^2(G)|$

$= 2m$ (m is odd). Conversely, if $p = 2$, $|\Delta^2(G)| = 2m$ (m is odd) and $J(kG)$ is nilpotent, then $J(kG)^2 = J(k\Delta^2(G))^2 kG = 0$. Thus, we obtain the next

Lemma. *The following are equivalent:*
(1) $J(kG) \neq 0$ *and* $J(kG)^2 = 0$.
(2) $p = 2$, $|\Delta^2(G)| = 2m$ *(m is odd) and* $J(kG)$ *is nilpotent.*

Now, a necessary and sufficient condition for $J(kG)$ to be nilpotent is given in [3, Corollary 8.1.14]. By making use of the similar argument as in the proof of it, we can obtain the next

Proposition 1. *The following are equivalent:*
(1) $J(kG) \neq 0$ *and* $J(kG)^2 = 0$.
(2) $p = 2$, $|\Delta^2(G)| = 2m$ *(m is odd) and* $kC_G(s)/\langle s \rangle$ *is semi-simple, where s is an involution of* $\Delta^2(G)$.

Proof. (1) $\Rightarrow$ (2) : By Lemma, it suffices to prove that $J(kH/\langle s \rangle) = 0$, where $H = C_G(s)$. Clearly $[G : H] < \infty$, and hence by [3, Lemma 7.2.8], $J(kH)$ is nilpotent. Thus, we have $J(kH) = N(kH) = J(k\Delta^2(H))kH$ ([3, Theorem 8.1.9]). Since $[G : H] < \infty$, clearly $\Delta^2(G) \supset \Delta^2(H)$, and therefore we have $\Delta^2(H) = \langle s \rangle$, because $|\Delta^2(G)| = 2m$. Thus, we get $J(kH) = J(k\langle s \rangle)kH$. This implies that $J(kH/\langle s \rangle) \cong J(kH)/J(k\langle s \rangle)kH = 0$.

(2) $\Rightarrow$ (1) : Since $J(kC_G(s))/J(k\langle s \rangle)kC_G(s) \cong J(kC_G(s)/\langle s \rangle) = 0$, we see that $J(kG_G(s)) = J(k\langle s \rangle)\, kC_G(s)$ is nilpotent. Thus because $[G : C_G(s)] < \infty$, $J(kG)$ is nilpotent by [3, Lemma 7.2.8], and so $J(kG)^2 = 0$ by Lemma.

Now, we assume that $p = 2$ and $|\Delta^2(G)| = 2m$ (m is odd). Then $J(k\Delta(G))^2 = 0$, because $J(k\Delta(G)) = J(k\Delta^2(G))k\Delta(G)$ ([3, Lemma 8.1.8]). Hence, if $[G : \Delta(G)] < \infty$, $J(kG)$ is nilpotent by [3, Lemma 7.2.8]. This together with Lemma implies the following

Proposition 2. *Assume that* $[G : \Delta(G)] < \infty$. *Then the following are equivalent:*
(1) $J(kG) \neq 0$ *and* $J(kG)^2 = 0$.
(2) $p = 2$ *and* $|\Delta^2(G)| = 2m$ *(m is odd).*

Next, suppose that $J(kG)^2 = 0$ and let N be a normal subgroup of $\Delta^2(G)$ of index 2 in $\Delta^2(G)$. Then N is a normal subgroup of G. If $\overline{G} = G/N$, then

$J(k\overline{G})^2 = 0$, because $k\overline{G}$ is isomorphic to a direct summand of kG. Noting that $\Delta^2(\overline{G}) = \Delta^2(G)/N \cong S$ a Sylow 2-subgroup of $\Delta^2(G)$, we get

$$J(k\overline{G}) = N(k\overline{G}) = J(k\Delta^2(\overline{G}))k\overline{G} = J(k\overline{S})k\overline{G},$$

where $\overline{S}$ is the image of S in $\overline{G}$. This together with the isomorphism $kG/\Delta^2(G) \cong k\overline{G}/\overline{S} \cong k\overline{G}/J(k\overline{S})k\overline{G}$ implies

$$J(kG/\Delta^2(G)) \cong J(k\overline{G}/J(k\overline{S})k\overline{G}) = 0.$$

Hence, we have the following

Proposition 3. *If* $J(kG) \neq 0$ *and* $J(kG)^2 = 0$, then $p = 2$, $|\Delta^2(G)| = 2m$ *(m is odd) and* $kG/\Delta^2(G)$ *is semi-simple.*

Unfortunately, up to the present, the structure of a group whose group algebra is semi-simple is not known, and so we cannot give the structure of $G/\Delta^2(G)$ in the above proposition. Accordingly, we don't know whether the converse of the proposition is true or not.

Now, we state our theorem as follows :

Theorem 1. *The following are equivalent :*

(1) $J(kG) \neq 0$ *and* $J(kG)^2 = 0$.

(2) $p = 2$ *and one of the following holds.*

(i) $|\Delta^2(G)| = 2$ *and* $J(kG/\Delta^2(G)) = 0$.

(ii) $|\Delta^2(G)| = 2m$, *where m is an odd number greater than* 1, *and the centralizer* $C_G(s)$ *of an involution s of* $\Delta^2(G)$ *has a subgroup H such that* $[C_G(s) : H] < \infty$ *and* $J(kH) = 0$.

Proof. (1) $\Rightarrow$ (2) : In view of Lemma and Proposition 3, we may assume that $p = 2$ and $|\Delta^2(G)| = 2m$, where m is an odd number greater than 1. Let s be an involution of $\Delta^2(G)$. Then it is evident from our assumption that $G \neq C_G(s)$. Hence we can choose an element x of $G - C_G(s)$. Now set $H = C_G(s) \cap C_G(s)^x$. Since $s \in \Delta^2(G)$, it is clear that $[G : C_G(s)] < \infty$, and so $[G : H] < \infty$. Thus we have $[C_G(s) : H] < \infty$. Since $J(kH)$ is nilpotent ([3, Lemma 7.2.8]), in order to prove that $J(kH) = 0$, it suffices to show that $\Delta^2(H) = \langle 1 \rangle$ ([3, Theorem 8.1.9]). Now, suppose that $\Delta^2(H)$ has an element $t \neq 1$ of order a power of 2. Then $t \in \Delta^2(G)$, because $[G : H] < \infty$. Further, since $t \in C_G(s)$ and $t \in C_G(s)^x$, it is clear that $\langle t, s \rangle$ and $\langle t, s^x \rangle$ are 2-subgroups of $\Delta^2(G)$. Hence we have $s = t = s^x$, because $|\Delta^2(G)| = 2m$. This implies that $x \in C_G(s)$ contrary to our choice of x.

This shows that $\Delta^2(H) = \langle 1 \rangle$.

(2) $\Rightarrow$ (1) : If (i) holds, then by Proposition 1, $J(kG)^2 = 0$. Next, suppose that (ii) holds. Then $\Delta^2(H) = \langle 1 \rangle$ by [3, Theorem 8.1.9]. Hence $H \cap \langle s \rangle = \langle 1 \rangle$, and so $T = H\langle s \rangle$ is a direct product of H and $\langle s \rangle$. Therefore, by [6, Lemma 2.8] $J(kT) = J(k\langle s \rangle)kT$, and hence $J(kT)^2 = 0$. Since $[G : T] < \infty$, $J(kG)$ is nilpotent. Thus, it follows from Lemma that $J(kG)^2 = 0$.

Corollary 1. *Let $p = 2$, $|\Delta^2(G)| = 2m$ (m is odd) and the center $Z(\Delta^2(G))$ of $\Delta^2(G) = \langle 1 \rangle$. Then $J(kG)^2 = 0$ if $kG/\Delta^2(G)$ is semi-simple.*

Proof. Set $H = C_G(\Delta^2(G))$. Then $[G : H] < \infty$, and so we have $[C_G(s) : H] < \infty$, where s is an involution of $\Delta^2(G)$. Set $\overline{G} = G/\Delta^2(G)$ and let $\overline{H}$ be the image of H in $\overline{G}$. Since $[\overline{G} : \overline{H}] < \infty$ and $J(k\overline{G}) = 0$, $J(k\overline{H})$ is nilpotent ([3, Lemma 7.2.8]), and so we have $J(k\overline{H}) = 0$ because $\Delta^2(\overline{H}) = \langle \overline{1} \rangle$. This together with the isomorphism

$$\overline{H} = H\Delta^2(G)/\Delta^2(G) \cong H/H \cap \Delta^2(G) = H/Z(\Delta^2(G)) \cong H$$

implies that $J(kH) = 0$. Hence $J(kG)^2 = 0$ by Theorem 1.

Remark. We assume that $J(kG) \neq 0$ and $J(kG)^2 = 0$. Let s be an involution of $\Delta^2(G)$. If $G \neq C_G(s)$, then $H_0 = \bigcap_{x \in G} C_G(s)^x$ is a normal subgroup of G of finite index in G because $[G : C_G(s)] < \infty$. Hence $J(kH_0)^2 = 0$ by [3, Theorem 7.2.7]. Therefore, noting that $\Delta^2(H_0) = \langle 1 \rangle$, we have $J(kH_0) = 0$. This shows that there exists a normal subgroup H_0 of G such that $J(kH_0) = 0$ and $[G : H_0]$ is even. Conversely, suppose that G has such a normal subgroup H_0 and $p = 2$. Besides, if we assume that $[G : H_0]$ is not divisible by 4, then G has a normal subgroup N such that $N \supset H_0$ and $[G : N] = 2$. Since $[N : H_0]$ is not divisible by 2, we have $J(kN) = J(kH_0)kN = 0$. Therefore it follows from [3, Theorem 7.2.7] that $J(kG)^2 = 0$. However, Wallace [6, Example 6.5] shows that there exists a group G with $J(kG)^2 = 0$ which contains a normal subgroup H_0 such that $[G : H_0] = 4$ and $J(kH_0) = 0$.

2. Let W be a finite group whose commutator subgroup is a p'-group. The genetators of $J(kW)$ have been given by Y. Tsushima [4]. Now, by making use of his result, we show the following

Theorem 2. *If the commutator subgroup G' of G has no elements of order a power of p, then $N(kG) = J(Z(kG))kG$.*

Proof. By [3, Theorem 8.1.9], $N(kG) = (\bigcup J(kW))kG$, where the union is over all finite normal subgroups W of G contained in $\Delta^p(G)$. Hence, in order to prove our theorem, it suffices to prove that ($*$) if W is a finite normal subgroup of G, then $J(kW) \subset J(Z(kG))kG$. From our assumption, it is easy to see that W is a p-nilpotent group with an abelian Sylow p-subgroup and $W' \subset W \cap G' \subset N$, where N is a normal p-complement in W. Now, let H be a subgroup of W containing N. Then it easily follows from $W \cap G' \subset N$ that H is normal in G. Hence the sum of the block idempotents of kH of full defect is an element of $Z(kG)$. Hence, in order to prove ($*$), by [4, Proposition 8], it suffices to prove that if C is a conjugate class of a p-element of W such that $C \subset H$, then C is a conjugate class in G. Let $g \in G$ and $s \in C$. Since $N\langle s \rangle$ is normal in G, there exist $x \in N$ and a positive integer n such that $gsg^{-1} = xs^nx^{-1}$. Hence we have $s^{n-1} \in G' \cap \langle s \rangle = \langle 1 \rangle$, and so $n = 1$. This implies that C is a conjugate class in G, and the theorem is proved.

If $G = \Delta(G)$, then $J(kG) = N(kG)$ by [3, Lemma 8.1.8], and so we have the following

Corollary 2. *If $G = \Delta(G)$ and G' has no elements of order a power of p, then $J(kG) = J(Z(kG))kG$.*

Remark. In [6, Example 6.3], D. A. R. Wallace has shown that there exists a group G which satisfies that $|G'| = 2$ and G/G' is a torsion-free abelian group. Now, let $p = 2$ and G a group with this property. Then we have

$$J(kG) = J(kG')kG = J(Z(kG))kG.$$

This implies that the converse of the corollary is not true.

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