Math. J. Okayama Univ. 25 (1983), 181—184

ON RINGS SATISFYING THE IDENTITY $(X-X^n)^2=0$

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Throughout, R will represent a ring with center C, and N the set of nilpotent elements in R. Let n be a positive integer greater than 1, and E_n the set of elements x in R such that $x=x^n$.

We consider the following properties:

(i) N is commutative.

(ii)$^*_n$ $(x-x^n)(y-y^n)=0$ for all $x, y\in R$.

(ii)$_n$ $(x-x^n)^2=0$ for all $x\in R$.

(ii)$'_n$ $(x-x^n)^n=0$ for all $x\in R$.

(iii)$_n$ Any $x\in R$ may be written in at most one way in the form $x=b+a$, where $b\in E_n$ and $a\in N$. (There may be elements x in R which cannot be written in the given form.)

If R satisfies (ii)$^*_n$ and (iii)$_n$, then R is called a *generalized n-ring*. Following [4], R is called a *generalized n-like ring* if $(xy)^n-xy^n-x^ny+xy=0$ for all $x, y\in R$, or equivalently, if $(x-x^n)(y-y^n)=0$ and $(xy)^n=x^ny^n$ for all $x, y\in R$ (see [4, Lemma 3]).

The major purpose of this paper is to prove the following

Theorem 1. *If R satisfies* (i), (ii)$_n$ *and* (iii)$_n$, *then R is commutative.*

In preparation for proving Theorem 1, we state the next lemma.

Lemma 1. (1) *Let R be a ring satisfying* (i) *and* (ii)$_n$. *Then N is a commutative nil ideal of bounded index at most 2. If there exists an integer $m>1$ such that $m^2x^4=0$ for all $x\in R$, then $x^{m^2}\in E_n$ for all $x\in R$.*

(2) *If R satisfies* (i) *and* (ii)$_n$, *then there exists a finite set P of prime numbers such that $R=\sum_{p\in P} R^{(p)}$, where $R^{(p)}=\{x\in R \mid px\in N\}$.*

(3) *Let R be a ring satisfying* (i), (ii)$_n$ *and* (iii)$_n$. *If there exists an integer $m>1$ such that $m^2x^4=0$ for all $x\in R$, then $[x^{m^2},a]=0$ for all $x\in R$ and $a\in N$.*

Proof. (1) By (ii)$_n$, there holds $x^{2n}=2x^{n+1}-x^2$. Hence, N is a commutative nil ideal of bounded index at most 2 by [2, Lemma 2 (2)]. Furthermore, an easy induction shows that $x^{\mu n-\mu+2}=\mu x^{n+1}-(\mu-1)x^2$, and so $x^{\mu n}=\mu(x^{n+\mu-1}-x^\mu)+x^\mu$ for any positive integer μ; in particular, $x^{m^2n}=$

$m^2(x^{n+m^2-1}-x^{m^2})+x^{m^2}=x^{m^2}$.

(2) Let $m=(2^n-2)^2$. Since N is an ideal of R by (1), we see that $(2^n-2)x=2^n(x-x^n)-\{2x-(2x)^n\}\in N$ for all $x\in R$, i.e., $m(R/N)=0$. As is well known, the factor ring R/N satisfying the polynomial identity $X-X^n=0$ is a subdirect sum of finite fields (see, e.g., [1, Theorem 19]). Noting here that $m(R/N)=0$, we can easily see the assertion. Needless to say, every $R^{(p)}$ is an ideal of R containing N.

(3) Let $x\in R$, and $a\in N$. According to (1), we have $(x+a)^{m^2}=x^{m^2}+a'+a''$, where $x^{m^2}\in E_n$, $a'=\sum_{i=0}^{m^2-1}x^{m^2-i-1}ax^i\in N$ and $a''\in N^2\subseteq C$. Since $(x+a)^{m^2}$ is also in E_n, (iii)$_n$ shows that $a'+a''=0$. Hence, $[x^{m^2},a]=[x,a']=[x,a']+[x,a'']=0$.

Proof of Theorem 1. In view of Lemma 1 (2), there exists a finite set P of prime numbers such that $R=\sum_{p\in P}R^{(p)}$, where $R^{(p)}$ is the ideal of R containing N defined by $\{x\in R \mid px\in N\}$. Obviously, (i), (ii)$_n$ and (iii)$_n$ are inherited by the ideal $R^{(p)}$. Since N is a nil ideal of bounded index at most 2 (Lemma 1 (1)), we see that $p^2x^2=0$ for all $x\in R^{(p)}$, and so $[x^{p^4},a]=0$ for all $x\in R^{(p)}$ and $a\in N$ (Lemma 1 (3)). As is well known, the factor ring $R^{(p)}/N$ satisfying the polynomial identity $X-X^n=0$ is a subdirect sum of finite fields of characteristic p, and hence we can find a positive integer k such that $x^{p^k}-x\in N$ for all $x\in R^{(p)}$. Now, let $x\in R^{(p)}$ and $a\in N$. Since $[x^{p^4},a]=0$ and $x^{p^{4k}}-x\in N$, we get $[x,a]=0$ by (i), which shows that N is in the center of $R^{(p)}$. Hence, N is contained in the center of R, and therefore R is commutative by [1, Theorem 19].

If R is a generalized n-ring, it is easy to see that $N^2=0$, and so N is commutative. Thus, as a direct consequence of Theorem 1, we have

Corollary 1. *Every generalized n-ring is commutative. In particular, every generalized n-like ring satisfying* (iii)$_n$ *is commutative.*

Corollary 2. *Suppose that there exists an integer* $m>1$ *such that* $(m,n-1)=1$ *and* $mN=0$. *Suppose that R satisfies* (i) *and* (ii)$_n$. *Then, R is commutative if and only if R satisfies* (iii)$_n$.

Proof. In view of Theorem 1, it suffices to show that if R is commutative then (iii)$_n$ is satisfied. Suppose that both b and $b+a$ are in E_n with some $a\in N$. Then $b+nab^{n-1}=(b+a)^n=b+a$ (Lemma 1 (1)), and so $nab^{n-1}=a$, whence it follows that $nab=nab^n=ab$. Hence, $na=n^2ab^{n-1}=nab^{n-1}=a$, namely $(n-1)a=0$. Since $ma=0$ and $(m,n-1)=1$, we get $a=0$, proving (iii)$_n$.

Next, motivated by [3, Theorem 1], we prove the following

Theorem 2. *Let p be a prime. If R satisfies* (i), (ii)$'_p$ *and $pR=0$, then the following are equivalent:*

1) *R is commutative.*
2) *R satisfies* (iii)$_p$.
3) *E_p is a subring of R.*
4) *E_p is an additive subgroup of R.*
5) *E_p is central.*

Proof. Obviously, $x^p \in E_p$ for any $x \in R$, and N is a commutative nil ideal of bounded index at most p by [2, Lemma 2 (2)]. Then, it is easy to see that 1)$\Rightarrow$3)$\Rightarrow$4)$\Rightarrow$2) and 1)$\Leftrightarrow$5).

2)$\Rightarrow$1). Let $x \in R$, and $a \in N$. Then we have $(x+a)^p = x^p + a' + a''$, where $x^p \in E_p$, $a' = \sum_{i=0}^{p-1} x^{p-i-1}ax^i \in N$ and $a'' \in N^2 \subseteq C$. Since $(x+a)^p$ is also in E_p, (iii)$_p$ shows that $a'+a''=0$. Hence, $[x^p,a]=[x,a']=[x,a']+[x,a'']=0$, and therefore $[x,a]=[x^p,a]+[x-x^p,a]=0$, which shows that $N \subseteq C$. Now, R is commutative by [1, Theorem 19].

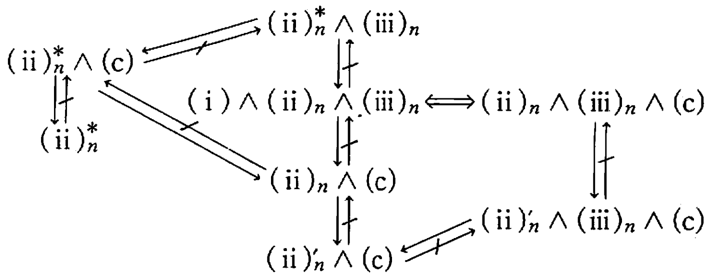
Examples. (1) The commutative ring $R=\boldsymbol{Z}/4\boldsymbol{Z}$ satisfies (ii)$_3^*$, but does not (iii)$_3$; the commutative ring $\boldsymbol{Z}/8\boldsymbol{Z}$ satisfies (ii)$'_3$, but does neither (ii)$_3$ nor (iii)$_3$.

(2) Let p be a prime. Then $R=\left\{\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \mid a, b \in \mathrm{GF}(p)\right\}$ is a non-commutative ring satisfying (ii)$_p^*$ and $pR=0$.

(3) Let $R=\left\{\begin{pmatrix} a & b & c \\ 0 & a & b \\ 0 & 0 & a \end{pmatrix} \mid a, b, c \in \mathrm{GF}(3)\right\}$. Then R is a commutative ring satisfying (ii)$'_3$ and $3R=0$, but not (ii)$_3$.

(4) Let $R=\left\{\begin{pmatrix} a & b & c & d \\ 0 & a & 0 & c \\ 0 & 0 & a & b \\ 0 & 0 & 0 & a \end{pmatrix} \mid a, b, c, d \in \mathrm{GF}(2)\right\}$. Then R is a commutative ring satisfying (ii)$_2$=(ii)$'_2$ and $2R=0$, but not (ii)$_2^*$.

These examples give the following table, where (c) signifies the property that R is commutative.



In conclusion, we would like to express our indebtedness and gratitude to the referee for his helpful suggestions and valuable comments.

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(*Received June 30, 1983*)