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SOME COMMUTATIVITY PROPERTIES FOR RINGS. II

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This is a natural sequel to the previous paper [7]. As for notations and terminologies used in this paper without mention, we follow [7]. A ring R is called *left* (resp. *right*) *s-unital* if $x \in Rx$ (resp. $x \in xR$) for each x in R, and R is called *s-unital* if R is both left and right s-unital.

Let A be a non-empty subset of the ring R with center C; let N denote the set of nilpotent elements of R, D the commutator ideal of R, and $V_R(A)$ the centralizer of A in R. Let q be a fixed integer greater than 1. We consider the following properties:

(I-A) For each $x \in R$, there exists a polynomial $f(\lambda)$ in $\boldsymbol{Z}[\lambda]$ such that $x - x^2 f(x) \in A$.

(I′-A) For each $x \in R$, either $x \in C$ or there exists a polynomial $f(\lambda)$ in $\boldsymbol{Z}[\lambda]$ such that $x - x^2 f(x) \in A$.

(II-A)$_q$ If $x, y \in R$ and $x - y \in A$, then either $x^q = y^q$ or x and y both belong to $V_R(A)$.

(III-A) For every $a \in A$ and $x \in R$, $[a,x]_2 = [[a,x],x] = 0$.

(III′-A) For every $a \in A$ and $x \in R$, there exists a positive integer $k = k(a,x)$ such that $[a,x]_k = [[a,x]_{k-1},x] = 0$.

(V) For every $x,y \in R$, there exists a positive integer $n = n(x,y)$ such that $[x,(xy)^n - (yx)^n] = 0 = [x,(xy)^{n+1} - (yx)^{n+1}]$.

(VI) For every $x, y, z \in R$, there exists a positive integer $n = n(x,y,z)$ such that $[x,(xyz)^n - y^n z^n x^n] = 0 = [x,(xyz)^{n+1} - y^{n+1} z^{n+1} x^{n+1}]$.

(VII) For every $x, y \in R$, there exist integers $n = n(x,y) \geq 1$ and $m = m(x,y) > 1$ such that $[x, x^n y - y^m x] = 0$.

(VII*) For each $y \in R$ there exists an integer $m = m(y) > 1$ such that $[x, x^n y - y^m x] = 0 = [x, x^n y^m - y^{m^2} x]$ for all $x \in R$, where n is a fixed positive integer.

(A)$_q$ If $a, a' \in A$ and $q[na',a] = 0$ for some positive integer n, then $[na',a] = 0$.

The major purpose of this paper is to enlarge the list of equivalent conditions in [7, Theorem 1] by adding six new conditions stated in the following theorem.

Theorem 1. *The following statements are equivalent*:

1) *R is commutative.*

2)′ *There exists a commutative subset A for which R satisfies* (I-A), (II-A)$_q$ *and* (III′-A).

5)′ *There exists a commutative subset A of N for which R satisfies* (I′-A) *and* (III′-A).

7) *R satisfies* (V) *and there exists a commutative subset A of N for which R satisfies* (I′-A).

8) *R satisfies* (VI) *and there exists a commutative subset A of N for which R satisfies* (I′-A).

9) *R satisfies* (VII) *and there exists a commutative subset A of N for which R satisfies* (I′-A).

9)* *R satisfies* (VII*) *and there exists a commutative subset A of N for which R satisfies* (I′-A).

We also reprove [6, Corollary 1] which improves the final theorem of Bell [2], and also give a generalization of the main theorem of Psomopoulos [5]. These results are included in the following

Theorem 2. *Let R be a left (or right) s-unital ring. Then the following statements are equivalent:*

1) *R is commutative.*

2) *There exists a subset A for which R satisfies* (I-A) *and* (II-A)$_2$.

3) *There exists a subset A for which R satisfies* (I-A), (II-A)$_q$ *and* (II-A)$_{q'}$, *where* q' *is an integer greater than* 1 *with* $(q, q')=1$.

4) *R satisfies* (VII) *and there exists a subset A of N for which R satisfies* (I′-A).

5) *R satisfies* (VII*).

In preparation for proving our theorems, we state the following two lemmas.

Lemma 1. (1) *Let* ψ *be a ring homomorphism of R onto* R^*. *If R satisfies* (I-A), (I′-A), (II-A)$_q$ *or* (III′-A), *then* R^* *satisfies* (I-$\psi(A)$), (I′-$\psi(A)$), (II-$\psi(A)$)$_q$ *or* (III′-$\psi(A)$), *respectively.*

(2) *If R satisfies* (II-A)$_q$, *then* $[a,x^q]=0$ *for all* $a\in A$ *and* $x\in R$.

(3) *If R satisfies* (I′-A) *and* (II-A)$_q$ (*resp.* (I′-A) *and* (III′-A)), *then R is normal in the sense that all the idempotents of R are central.*

(4) *If A is commutative and R satisfies* (I′-A), *then N is a commutative nil ideal of R containing D and is contained in* $V_R(A)$; *in particular,* $N^2\subseteq C$.

(5) *Let R be a normal, subdirectly irreducible ring. If A is a commutative subset of N not contained in C for which R satisfies* (I$'$-A), *then R is of characteristic p^α,* where *p is a prime and $\alpha > 0$. When this is the case, $\bar{b}=b+N(\in R/N)$ is algebraic over* GF(p) *provided $b\in R\backslash V_R(A)$ (see (4)).*

(6) *Let R be an s-unital ring. If R satisfies* (II-A)$_q$ *and* $(A)_q$, *then A is commutative.*

Proof. (1) Straightforward.

(2) This is [7, Lemma 1 (3)].

(3) See the proof of [7, Lemma 1 (4)].

(4) This is [7, Lemma 1 (5)].

(5) See the proof of [7, Lemma 2].

(6) Suppose $[a,b]\neq 0$ for some $a,b\in A$. As is well known, there exists $e\in R$ such that $ae=ea=a$. Then, by (II-A)$_q$, $(a+e)^q=e^q$ and hence $qa\in a\langle a\rangle$. Also, by (II-A)$_q$, $a^q=0$ and hence $q^{q-1}a\in a\langle a\rangle^{q-1}=0$. Hence, there exists a positive integer m such that $[q^m a,b]=0$ and $[q^{m-1}a,b]\neq 0$. But, this is impossible by $(A)_q$.

Lemma 2. (1) *If R satisfies* (V) (*resp.* (VII)), *then R is normal.*

(2) *If R satisfies* (VI), *then R is normal.*

(3) *Let R be a left* (*or right*) *s-unital ring. If R satisfies* (VII), *then $N\subseteq C$.*

Proof. (1) Given an idempotent e and an element x in R, there exists an integer $n=n(e,e+ex(1-e))\geq 1$ (resp. $n=n(e,e+ex(1-e))\geq 1$ and $m=m(e,e+ex(1-e))>1$) such that

$$ex(1-e)=[e,\{e(e+ex(1-e))\}^n-\{(e+ex(1-e))e\}^n]=0$$
$$(\text{resp. } ex(1-e)=[e,e^n(e+ex(1-e))-(e+ex(1-e))^m e]=0).$$

Hence, $ex=exe$, and similarly $xe=exe$.

(2) Given an idempotent e and an element x in R, there exists a positive integer $n=n(e,e,e+ex(1-e))$ such that

$$ex(1-e)=[e,\{e\cdot e(e+ex(1-e))\}^n-e^n(e+ex(1-e))^n e^n]=0.$$

Hence, $ex=exe$, and similarly $xe=exe$.

(3) Let $a\in N$ and $x\in R$. By hypothesis, there exist integers $n_1=n(x,a)\geq 1$ and $m_1=m(x,a)>1$ such that $x^{n_1}[x,a]=[x,a^{m_1}]x$. Next, choose $n_2=n(x,a^{m_1})\geq 1$ and $m_2=m(x,a^{m_1})>1$ such that $x^{n_2}[x,a^{m_1}]=[x,a^{m_1 m_2}]x$, and so on. Then, for any positive integer t we have

$$x^{n_1+n_2+\cdots+n_t}[x,a]=[x,a^{m_1m_2\cdots m_t}]x^t.$$

Since a is nilpotent, $x^{n_1+n_2+\cdots+n_t}[x,a]=0$ for sufficiently large t. Then, if R contains 1, the usual argument of replacing x by $x+1$, etc. shows that $[x,a]=0$.

We claim here that R is s-unital. Let $a\in N$. Since R is left s-unital, choose $e\in R$ with $ea=a$. Then, by the above, we can easily see that $a-ae=e^{\nu}[e,a]=0$ with some ν. Now, let x be an arbitrary element of R, and choose $e'\in R$ with $e'x=x$. Then, as is well known, there exists $e''\in R$ such that $e''x=x$ and $e''e'=e'$. Since $(x-xe'')^2=0$ and $e'(x-xe'')=x-xe''$, the fact just claimed above implies that $x-xe''=(x-xe'')e'=0$, which proves that R is s-unital. Thus, in view of [4, Proposition 1], we may assume that R has 1, and therefore $N\subseteq C$.

Proof of Theorem 1. Obviously, 1) implies 2)′−9)*, and 9)* does 9).

2)′⇒1). In view of Lemma 1 (1), we may, and shall, assume that R is subdirectly irreducible. According to [3, Theorem 19] and (I-A), it suffices to show that $A\subseteq C$. Suppose, to the contrary, that there exist $a\in A$ and $b\in R$ such that $[a,b]_1=[a,b]\neq 0$. Then, by (III′-A), $[a,b]_{k-1}\neq 0$ and $[a,b]_k=0$ for some $k>1$. By the proof 2)⇒1) of [7, Theorem 1], we see that R contains 1, $p^{\alpha}[a,b]=0$ and $[a,b^{t^{\gamma}}]=0$, where γ, $\alpha>0$ and $(p,t)=1$. Hence, $[[a,b]_{k-2},b],b]=[a,b]_k=0$ yields $t^{\gamma}b^{t^{\gamma}-1}[a,b]_{k-1}=[[a,b]_{k-2},b^{t^{\gamma}}]=0$, where $[a,b]_0=a$. Then, the usual argument of replacing b by $b+1$, etc. shows that $t^{\gamma}[a,b]_{k-1}=0$. Since $p^{\alpha}[a,b]_{k-1}=0$ and $(p,t)=1$, it follows a contradiction $[a,b]_{k-1}=0$.

5)′⇒1). Again, in view of Lemma 1 (1), we may assume that R is subdirectly irreducible. According to [3, Theorem 19] and (I′-A), it suffices to show that $A\subseteq C$. Suppose, to the contrary, that there exist $a\in A$ and $b\in R$ such that $[a,b]_{k-1}\neq 0$ and $[a,b]_k=0$ for some $k>1$. Then, by Lemma 1 (3),(4) and (5), R is of characteristic p^{α} (p a prime and $\alpha>0$), and $\bar{b}=b+N$ is algebraic over GF(p). Furthermore, noting that every non-zero idempotent of R/N coincides with 1 (Lemma 1 (3)), we can easily see that $\langle\bar{b}\rangle=\mathrm{GF}(p^{\beta})$ with some $\beta>0$, and therefore $b^{p^{\gamma}}-b\in N$ for some $\gamma\geq\alpha$. Hence, $[[a,b]_{k-2},b^{p^{\gamma}}-b]=0$ by Lemma 1 (4), and $[[a,b]_{k-2},b^{p^{\gamma}}]=p^{\gamma}b^{p^{\gamma}-1}[a,b]_{k-1}=0$ by $[a,b]_k=0$. From these we get

$$[a,b]_{k-1}=[[a,b]_{k-2},b^{p^{\gamma}}]-[[a,b]_{k-2},b^{p^{\gamma}}-b]=0.$$

This contradiction proves that R is commutative.

7) (resp. 8))⇒1). Again, in view of Lemma 1 (1), we may assume that R is subdirectly irreducible. Let x be an arbitrary element in $R\backslash C$.

By (I$'$-A), there exists $y \in \langle x \rangle$ such that $x^m = x^{m+1}y$ with some positive integer m. Obviously, $e = x^m y^m$ is a central idempotent with $x^m = x^m e$ (Lemma 2 (1)) (resp. (2)), and e is either 0 or 1. If $e=0$ then x is in the commutative ideal N, and so $[[a,x],x]=0$ for all $a \in A$ (Lemma 1 (4)). On the other hand, if $e=1$ then x is a unit. Now, let u be an arbitrary unit in R. For any $a \in A$, there exists a positive integer $n = n(u^{-1}, u(1+a))$ (resp. $n = n(u, 1+a, u^{-1})$) such that

$$[u^{-1}, (1+a)^n - u(1+a)^n u^{-1}] = [u^{-1}, (u^{-1}u(1+a))^n - (u(1+a)u^{-1})^n] = 0$$
$$(\text{resp. } [u,[u,(1+a)^n]]u^{-1} = [u,(u(1+a)u^{-1})^n - (1+a)^n u^{-n} u^n] = 0).$$

Noting here that $a^k \in C$ for all $k \geq 2$ (Lemma 1 (4)), we get

$$nu^{-1}[[a,u],u]u^{-2} = n[u^{-1}, a - uau^{-1}] = 0 \text{ (resp. } n[u,[u,a]]u^{-1} = 0),$$

and therefore $n[[a,u],u]=0$. Similarly, we have $(n+1)[[a,u],u]=0$. From these, we obtain $[[a,u],u]=0$. If x is in C, then $[[a,x],x]=0$ trivially. We have thus seen that R satisfies (III-A). Hence R is commutative by 5)$'$.

9) ⇒ 1). Careful scrutiny of the preceding proof shows that it remains only to prove that if R is a subdirectly irreducible ring with 1 then $[[a,x],x] = 0$ for all $a \in A$ and $x \in R$. However, in Lemma 2 (3), we have seen that $A \subseteq C$.

Theorem 1 includes obviously the main theorem of [1], and the next is an easy combination of Lemma 1 (6) and Theorem 1 2)$'$.

Corollary 1. *Let R be an s-unital ring. If there exists a subset A for which R satisfies* (I-A), (II-A)$_q$, (III$'$-A) *and* $(A)_q$, *then R is commutative.*

Proof of Theorem 2. Obviously, 1) implies 2)—5).

2)⇒1). We claim first that if R has 1 then R is commutative. Suppose $[b,c] \neq 0$ for some $b, c \in A$. Then, by (II-A)$_2$, we have $(b+c)^2 = b^2 = 0 = c^2$ and $2b = (b+1)^2 - 1 = 0$. Hence, $[b,c] = bc + cb = 0$. This contradiction shows that A is commutative. Now, suppose that $[a,x] \neq 0$ for some $a \in A$ and $x \in R$. Since $(x+a)^2 = x^2$ and $(x+1+a)^2 = (x+1)^2$ by (II-A)$_2$, we have

$$2a = \{x^2 + 2(x+a) + 1\} - (x+1)^2 = (x+1+a)^2 - (x+1)^2 = 0,$$

and therefore $[[a,x],x] = [a,x^2] + 2x^2 a - 2xax = 0$ by Lemma 1 (2). We have thus seen that R satisfies (III-A), and R is commutative by Theorem 1 2)$'$.

We now proceed to prove the general case. In view of Lemma 1 (1), we may assume that $R(\neq 0)$ is subdirectly irreeucible. If there exists a non-nilpotent element not contained in $V_R(A)$, (I-A) and (II-A)$_2$ together with Lemma 1 (3) show that the subdirectly irreducible ring R has 1, and therefore R is commutative by the above claim. We assume henceforth that $R\backslash N\subseteq V_R(A)$. Suppose $[b,c]\neq 0$ for some b, $c\in A$, and choose $e\in R$ with $eb=b$. Since e is not nilpotent, there holds $be=eb=b$ and $ce=ec$. Then, the argument employed in the proof of the above claim applies to see that $[b,c]=0$. This contradiction shows that A is commutative, and therefore $N\subseteq V_R(A)$ by Lemma 1 (4). This together with $R\backslash N\subseteq V_R(A)$ implies $R=V_R(A)$, i.e., $A\subseteq C$. Hence, R is commutative by [3, Theorem 19].

3)$\Rightarrow$1). In view of Lemma 1 (1), we may assume that R is subdirectly irreducible.

We claim first that if R has 1 then R is commutative. Suppose $[b, c]\neq 0$ for some b, $c\in A$. Then, by $(\text{II-}A)_q$ and $(\text{II-}A)_{q'}$, we have $q^{q-1}b=0=q'^{q'-1}b$ (see the proof of Lemma 1 (6)). Hence, $(q, q')=1$ yieds $b=0$. This contradiction shows that A is commutative, and therefore $N\subseteq V_R(A)$ by Lemma 1 (4). Now, suppose that $[x,a]\neq 0$ for some $x\in R\backslash N$ and $a\in A$. Then, by (I-A), $x-x^2y\in A$ for some $y\in\langle x\rangle$. And so, $x^q=x^{2q}y^q$ by $(\text{II-}A)_q$. As is easily seen, x^qy^q is a non-zero central idempotent (Lemma 1 (3)), i.e., $x^qy^q=1$, which shows that x is a unit of R. Now, by $(\text{II-}A)_q$ and $(\text{II-}A)_{q'}$, we have $(x+a)^q=x^q$ and $(x+a)^{q'}=x^{q'}$. Since x and $x+a$ are units in R and $(q, q')=1$, this forces a contradiction $x+a=x$, i.e., $a=0$. We have thus seen that $R\backslash N\subseteq V_R(A)$. Combining this with $N\subseteq V_R(A)$, we get $R=V_R(A)$, i.e., $A\subseteq C$. Hence, R is commutative by (I-A) and [3, Theorem 19].

We can now apply the argument used in the latter part of the proof of 2)$\Rightarrow$1) to get the conclusion.

4)$\Rightarrow$1). Since $A\subseteq C$ by Lemma 2 (3), R is commutative by [3, Theorem 19] and (I′-A).

5)$\Rightarrow$1): As was claimed in the proof of Lemma 2 (3), R is s-unital. So, in view of [4, Proposition 1], we may assume that R has 1. By hypothesis, we have

(*) $\quad x^n[x,y]=[x,y^m]x$ and $x^n[x,y^m]=[x,y^{m^2}]x$, where $m=m(y)$.

We claim here that $D\subseteq N$. By (*), we see that

$$(x+1)^n[x,y]x=[x,y^m](x+1)x=x^n[x,y](x+1) \text{ for all } x,\ y\in R.$$

But, $x=E_{22}$ and $y=E_{21}+E_{22}$ fails to satisfy $(x+1)^n[x,y]x-x^n[x,y](x+1)=0$ in $(\mathrm{GF}(p))_2$ (p a prime). Hence, $D\subseteq N$ by [4, Proposition 2]. Combining this with Lemma 2 (3), we get $D\subseteq C$.

If $n=1$, we obtain

$$[x,y-y^m]=\{(x+1)[x,y]-[x,y^m](x+1)\}-\{x[x,y]-[x,y^m]x\}=0$$

for all $x\in R$. Thus, R is commutative by [3, Theorem 19]. So we assume henceforth that $n>1$. We set $j=2^{n+1}-2^2$ (>0). Then, by (*), $jx^n[x,y]=(2x)^n[2x,y]-[2x,y^m]2x=0$, and so the usual argument of replacing x by $x+1$, etc. shows that $j[x,y]=0$. We obtain therefore, by $D\subseteq C$, $[x^j,y]=jx^{j-1}[x,y]=0$, i.e., $x^j\in C$ for all $x\in R$. Furthermore, using (*) and $D\subseteq C$ several times, we see that

$$\begin{aligned}(1-y^{(m-1)^2})[x,y]x^{2n-1}&=[x,y^m]x^n-y^{(m-1)^2}[x,y^m]x^n\\&=[x,y^m]x^n-my^{m(m-1)}[x,y^m]x\\&=[x,y^m]x^n-[x,y^{m^2}]x=0.\end{aligned}$$

Thus, $(1-y^{(m-1)^2})[x,y]x^{2n-1}=0$. Again, the usual argument of replacing x by $x+1$, etc. in the last identity shows that $(1-y^{(m-1)^2})[x,y]=0$. Hence, since $y^j\in C$, we get

$$[x,y-y^{j(m-1)^2+1}]=(1-y^{j(m-1)^2})[x,y]=0.$$

This proves that $y-y^{j(m-1)^2+1}\in C$ for $m=m(y)$, and therefore R is commutative by [3, Theorem 19].

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