Math. J. Okayama Univ. 25 (1983), 165—171

NOTE ON STRONGLY M-INJECTIVE MODULES

SHOJI MORIMOTO

Given a unitary left module M over a ring R with identity, G. Azumaya [1] introduced the useful notion of M-projective and M-injective modules; the notion of a strongly M-projective module has subsequently been given by K. Varadarajan [8]. In this note, we investigate strongly M-injective modules. Let t be the left exact preradical (for R-mod) corresponding to the left linear topology which has the smallest element $l_R(M)$, the left annihilator of M. We prove first that a strongly M-injective module is nothing but a t-weakly divisible module (Theorem 1.3); in particular, if M is faithful then every strongly M-injective module is injective which corresponds to [1, Theorem 14]. Next, we give the conditions for t to be a radical and to be stable (Theorems 2.4 and 2.7).

Throughout this note, R denotes a ring with identity, and modules mean unitary left R-modules, unless otherwise specified. The category of all modules is denoted by R-mod, and the injective hull of $A \in R$-mod by $E(A)$. As for terminologies and basic properties concerning preradicals and torsion theories, we refer to [6]. If r is a left exact preradical, then $L(r)=\{{}_RI \subseteq {}_RR \mid r(R/I)=R/I\}$ is a left linear topology on R. As is well-known, $L(r)$ has the smallest element if and only if $T(r)$ is closed under direct products (see, [3. III. 2. E4]).

In what follows, we fix a module M and denote by t the left exact preradical corresponding to the left linear topology which has the smallest element $T=l_R(M)$. As is easily seen, $t(A)=r_A(T)=\{a \in A \mid Ta=0\}$ $(A \in R\text{-mod})$, and so A is t-torsion if and only if $TA=0$.

1. A module Q is called *strongly M-injective* if every homomorphism of any submodule of M into Q can be extended to a homomorphism of M^J into Q for any index set J. Clearly, every injective module is strongly M-injective and every strongly M-injective module is M-injective. But the converse is not necessarily true (see Examples 1.6 and 1.8). Also, a direct product of modules is strongly M-injective if and only if so are all its factors.

Now, let r be a preradical. A module Q is said to be *r-weakly divisible* (resp. *r-divisible*) if for any exact sequence in R-mod $0 \to A \to B \to C \to 0$ with B (resp. C) r-torsion, $\mathrm{Hom}_R(-,Q)$ preserves the exactness. Since M is t-torsion, every t-weakly divisible module is strongly M-injective.

Lemma 1.1. *Let Q be a module. Then there holds the following*:
(1) *Q is strongly M-injective if and only if so is $t(Q)$.*
(2) *Q is t-weakly divisible if and only if so is $t(Q)$.*

Proof. As the proofs of the both are quite similar, we prove only (1). Suppose first that Q is strongly M-injective, and consider a row-exact diagram in R-mod

$$\begin{array}{ccccc} 0 & \longrightarrow & N & \xrightarrow{\ i\ } & M^J \\ & & {\scriptstyle f}\downarrow & & \\ & & t(Q) & & \\ & & {\scriptstyle j}\downarrow & & \\ & & Q & & \end{array}$$

where j is the inclusion. Then there exists $g: M^J \to Q$ such that $g \circ i = j \circ f$, and $g(M^J) = g(t(M^J)) \subseteq t(Q)$. Thus $t(Q)$ is strongly M-injective. Conversely, suppose that $t(Q)$ is strongly M-injective and consider a row-exact diagram in R-mod

$$\begin{array}{ccccc} 0 & \longrightarrow & N & \xrightarrow{\ i\ } & M^J \\ & & {\scriptstyle f}\downarrow & & \\ & & Q & & \end{array}$$

Since N is t-torsion, $f(N) \subseteq t(Q)$, and therefore, by assumption, $g \circ i = f$ with some $g: M^J \to t(Q)$. Thus Q is strongly M-injective.

Lemma 1.2. *The following are equivalent for a module Q*:
1) *$t(Q)$ is strongly M-injective.*
2) *$t(Q)$ is t-weakly divisible.*
3) *$t(Q)$ is injective as a left R/T-module.*

Proof. Obviously, 2)$\Rightarrow$1).
1)$\Rightarrow$3). Consider a row-exact diagram in R/T-mod

$$\begin{array}{ccccc} 0 & \longrightarrow & I/T & \xrightarrow{\ i\ } & R/T \\ & & {\scriptstyle f}\downarrow & & \\ & & t(Q) & & \end{array}$$

Since R/T can be embedded into a direct product of copies of M and $t(Q)$ is strongly M-injective by assumption, there exists an R-homomorphism $g: R/T \longrightarrow t(Q)$ such that $g \circ i = f$.
3)$\Rightarrow$2). Consider a row-exact diagram in R-mod

$$\begin{array}{ccccc} 0 & \longrightarrow & A & \xrightarrow{\ i\ } & B \\ & & {\scriptstyle f}\downarrow & & \\ & & t(Q) & & \end{array}$$

where B is t-torsion. Then, regarding the above diagram as in R/T-mod, we find an R/T-homomorphism $g: B \rightarrow t(Q)$ such that $g \circ i = f$, and hence $t(Q)$ is t-weakly divisible.

Now, combining Lemmas 1.1 and 1.2, we readily obtain

Theorem 1.3. *The following conditions are equivalent for a module Q:*
1) *Q is strongly M-injective.*
2) *Q is t-weakly divisible.*
3) *$t(Q)$ is injective as a left R/T-module.*

In particular, if every t-torsion module is strongly M-injective, then every R/T-module is injective, and hence R/T is semisimple artinian.

As is easily seen, M is faithful if and only if $t=1$. Thus we have

Corollary 1.4. *If M is faithful, the following are equivalent:*
1) *Q is strongly M-injective.*
2) *Q is t-divisible.*
3) *Q is injective.*

Corollary 1.5. *Let Q be a module with $E(Q)$ t-torsion. Then the following are equivalent:*
1) *Q is strongly M-injective.*
2) *Q is t-divisible.*
3) *Q is injective.*

Proof. It suffices to show that 1) $\Rightarrow$ 3). Since Q is t-weakly divisible and $E(Q)$ is t-torsion, Q is a direct summand of $E(Q)$. Hence $Q=E(Q)$.

We give here an example of a strongly M-injective module which is not injective.

Example 1.6. Let K be a field, and $R=\begin{pmatrix} K & K \\ 0 & K \end{pmatrix}$. Then $M=\begin{pmatrix} K & K \\ 0 & 0 \end{pmatrix}$ is an idempotent two-sided ideal of R. Now, we let (T_1, T_2, T_3) be the 3-fold torsion theory corresponding to M (see, e.g. [4]). Since R/M is a flat right R-module, T_1 is a TTF-class. Let t be the torsion functor corresponding to (T_1, T_2). Then, as is well-known, the left Gabriel topology corresponding to t has the smallest element $l_R(M)$. Since $E(R)=\begin{pmatrix} K & K \\ K & K \end{pmatrix}$, M coincides with $t(E(R))$ and is strongly M-injective. However, M is not injective, since it is not a direct summand of ${}_RR$.

Combining Lemma 1.1 and [7, Theorem 2.8] with the latter part of Theorem 1.3, we readily obtain

Corollary 1.7. *The following conditions are equivalent:*

1) *Every module is strongly M-injective.*
2) *Every t-torsion module is strongly M-injective.*
3) *R/T is semisimple artinian.*
4) *M^J is completely reducible for any index set J.*
5) *Every cyclic module is strongly M-projective.*
6) *Every module is strongly M-projective.*

Example 1.8. Let M be a field, and $R=M^J$, where J is an arbitrary infinite set. Then M is a simple R-module. Clearly, M^J is not completely reducible. In view of Corollary 1.7, this shows that there exists an M-injective module which is not strongly M-injective.

2. Given a left exact preradical r, we define the following left exact preradicals r' and r_2 (for R-mod):

$$r'(A)=\{a\in A \mid IJa=0 \text{ for some } I \text{ and } J \text{ in } L(r)\};$$
$$r_2(A)/r(A)=r(A/r(A)).$$

Lemma 2.1. (1) $r(A)\subseteq r'(A)\subseteq r_2(A)$.

(2) *If $a\in r_2(A)$, then $SIa=0$ for some $I\in L(r)$, where $S=\bigcap_{J\in L(r)} J$.*

(3) *$L(r')=\{{}_RK\subseteq {}_RR \mid K\supseteq IJ$ for some I and J in $L(r)\}$.*

(4) *$r=r'$ if and only if $L(r)$ is closed under finite products.*

(5) *If $L(r)$ has the smallest element S, then $L(r_2)$ has the smallest element S^2.*

(6) *$r=r_2$ if and only if r is a radical.*

(7) *If $L(r)$ has the smallest element, then $r'=r_2$.*

Proof. We can easily see (2)—(4) and (6). Moreover, (5) and (7) follow from (2). It suffices therefore to prove (1). Clearly, $r(A)\subseteq r'(A)$ for any $A\in R$-mod. If $x\in r'(A)$, then there exist I and J in $L(r)$ such that $IJx=0$. Since $r(A)=\{y \in A \mid l_R(y) \in L(r)\}$, we have $Jx \subseteq r(A)$. Thus, $J\bar{x}=0$, where $\bar{x}=x+r(A) \in r'(A)/r(A)$. This means $\bar{x} \in r(r'(A)/r(A)) \subseteq r(A/r(A))$, and therefore $x \in r_2(A)$.

The next shows that (1), (2) and (3) in [5, Theorem 6] are still equivalent for $n=1$.

Corollary 2.2 ([3, III. 2. E5]). *t is a radical if and only if $T=T^2$.*

Proposition 2.3. *If r is a left exact preradical, then the following are equivalent*:

1) *r is a radical.*
2) *Every r-weakly divisible module is r_2-weakly divisible.*
3) *Every r-torsion r-weakly divisible module is r-divisible.*

Proof. Obviously, 1) implies 2).

2) $\Rightarrow$ 3). Let A be an r-torsion r-weakly divisible module. Then $A=r(A)=r(E(A))$. Since A is r_2-weakly divisible by assumption, it is a direct summand of $r_2(E(A))$. On the other hand, it is essential in $r_2(E(A))$, and so $A=r_2(E(A))$. Hence, $r(E(A)/A)=r(E(A)/r(E(A)))=r_2(E(A))/r(E(A))=0$, proving that A is r-divisible.

3) $\Rightarrow$ 1). Let A be a module. Since $r(E(A))$ is r-divisible by assumption, [7, Proposition 1.2] shows that

$$\begin{aligned} 0&=(r(E(A))+r(E(A)))/r(E(A))=r(E(A)/r(E(A))) \\ &\supseteq r(A+r(E(A))/r(E(A))) \simeq r(A/(A \cap r(E(A)))=r(A/r(A)), \end{aligned}$$

proving that r is a radical.

Combining Proposition 2.3 with Corollary 2.2, we get at once

Theorem 2.4. *The following are equivalent*:

1) *Every t-weakly divisible module is t_2-weakly divisible.*
2) *Every t-torsion t-weakly divisible module is t-divisible.*
3) *t is a radical.*
4) $T=T^2$.

A preradical r is said to be *stable* if $T(r)$ is closed under injective hulls. As another application of Proposition 2.3, we have

Lemma 2.5 ([3, Proposition I.3.2]). *Every stable left exact preradical r is a radical.*

Proof. If A is an r-torsion r-weakly divisible module, then $A=r(A)=r(E(A))=E(A)$. Hence A is r-divisible, and therefore r is a radical by Proposition 2.3.

Proposition 2.6. *Let r be an idempotent preradical. If every r-torsion r-weakly divisible module is injective, then r is stable, and conversely.*

Proof. Let $A\in T(r)$. Since $r(E(A))$ is injective by assumption, it is a direct summand of $E(A)$. Furthermore, it is essential in $E(A)$, and therefore $r(E(A))=E(A)$. The converse is clear.

Now, we can prove the next

Theorem 2.7. *The following conditions are equivalent:*

1) *Every t-torsion strongly M-injective module is injective.*
2) *Every injective left R/T-module is injective.*
3) *t is stable.*
4) *t is a radical and R/T is flat as a right R-module.*
5) *$T=T^2$ and R/T is flat as a right R-module.*
6) *$(T(t)^l, T(t), F(t))$ is a hereditary 3-fold torsion theory, where $T(t)^l=\{A\in R\text{-mod} \mid \mathrm{Hom}_R(A,X)=0$ for all $X\in T(t)\}$.*

Proof. Obviously, 4)$\Rightarrow$5), and it is well-known that 5)$\Leftrightarrow$6)$\Rightarrow$3). The equivalence of 1)—3) is immediate by Lemma 1.2 and Proposition 2.6. Furthermore, 3) and 4) are equivalent by Lemma 2.5 and [2, Theorem 6].

Acknowledgement. The author would like to thank the referee and Professor Y. Kurata for their helpful suggestions.

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HAGI KŌEN GAKUIN HIGH SCHOOL

(*Received May 6, 1983*)