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$\aleph_0$-CONTINUOUS DIRECTLY FINITE PPOJECTIVE MODULES OVER REGULAR RINGS

Dedicated to Prof. Kentaro Murata on his 60th birthday

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We have defined, in [5], the notions of quasi-$\aleph_0$- and $\aleph_0$-continuous modules which can be regarded as generalized notions of right $\aleph_0$-continuous regular rings (see [2], [3]). In this paper, we study the properties of quasi-$\aleph_0$-continuous directly finite projective modules over a regular ring.

Let R be a regular ring. For a quasi-$\aleph_0$-continuous directly finite projective R-module M and essentially $\aleph_0$-generated submodules A and B of M with $A \cong B$, it is shown that $A^* \cong B^*$, where A^* and B^* are direct summands of M with $A \leq_e A^*$ and $B \leq_e B^*$, respectively (Theorem 5). This is a generalization of [2, Corollary 14.26]. Using this theorem, we show that if M and N are quasi-$\aleph_0$-continuous directly finite projective R-modules then $M \oplus N$ is directly finite (Theorem 8). Finally, we shall give examples of quasi-$\aleph_0$-continuous directly finite projective modules over regular rings which are not finitely generated.

Throughout this paper, R is a ring with identity and R-modules are unitary right R-modules. If M and N are R-modules, then the notation $N \lesssim M$ means that N is isomorphic to a submodule of M. For a submodule N of an R-module M, $N \leq_e M$ means that N is essential in M, while $N <\oplus M$ means that N is a direct summand of M.

Let M be an R-module and let $\mathscr{A}(M)$ be the family of all submodule A of M such that A contains a countably generated essential submodule. Given such an $\mathscr{A}(M)$, we consider the following conditions:

(C_1) For any $A \in \mathscr{A}(M)$ there exists a submodule A^* of M such that $A \leq_e A^*$ and $A^* <\oplus M$.

(C_2) For any $A \in \mathscr{A}(M)$ with $A <\oplus M$, any exact sequence $0 \to A \to M$ splits.

(C_3) For any $A \in \mathscr{A}(M)$ with $A <\oplus M$, if $N <\oplus M$ and $A \cap N = 0$ then $A \oplus N <\oplus M$.

We say that M is *quasi-$\aleph_0$-continuous* (resp. *$\aleph_0$-continuous*) if M satisfies the conditions (C_1) and (C_3) (resp. (C_1) and (C_2)). According to [8], every quasi-injective module is $\aleph_0$-continuous, every $\aleph_0$-continuous module is quasi-$\aleph_0$-continuous and M is quasi-$\aleph_0$-continuous if and only if M satisfies (C_1) and the condition

($*$) For any $A \in \mathscr{A}(M)$ and a direct summand N of M with $A \cap N = 0$, every homomorphism from A to N can be extended to a homomorphism from M to N.

Let M be an R-module. A submodule B of M is said to be *$\mathcal{S}$-closed* if M/B is nonsingular. For any submodule A of M there exists the smallest $\mathcal{S}$-closed submodule C of M containing A, which is called the *$\mathcal{S}$-closure* of A in M.

Now, we recall the following

Lemma 1 ([5, Lemma 1]). *Let M be an R-module, and let A and B be submodules of M such that $A \leq_e B$. Then B is contained in the $\mathcal{S}$-closure of A in M. If, in addition, M is nonsingular and $B <\oplus M$, then B coincides with the $\mathcal{S}$-closure of A in M.*

We note that if M is a nonsingular module satisfying (C_1) then, for any $A \in \mathcal{A}(M)$, there exists a unique A^* such that $A \leq_e A^*$ and $A^* <\oplus M$. Moreover, if M is nonsingular and quasi-$\aleph_0$-continuous (resp. $\aleph_0$-continuous), then so is every direct summand of M (Lemma 1).

Lemma 2. *Let M be a quasi-$\aleph_0$-continuous R-module, and let A, $B \in \mathcal{A}(M)$. If $A \cap B=0$ and $A \cong B$, then $A^* \cong B^*$.*

Proof. Let $f: A \rightarrow B$ be an isomorphism. Since $A \cap B^*=0$, in view of (*), f can be extended to a homomorphism $f^*: A^* \rightarrow B^*$, which is a monomorphism, because $A \leq_e A^*$. Since $A^* \cong f^*(A^*) \in \mathcal{A}(M)$, $f^*(A^*) \leq_e B^*$ and $A^* \cap f^*(A^*)=0$, again by (*), $(f^*)^{-1}: f^*(A^*) \rightarrow A^*$ can be extended to a monomorphism $h: B^* \rightarrow A^*$. Then h is an isomorphism, and thus $f^*=h^{-1}$ is an isomorphism with $f^*|_A=f$.

An R-module M is *directly finite* provided that M is not isomorphic to any proper direct summand of itself. We can use Lemma 2 to get the following generalization of [5, Theorem 2]. The proof is quite similar to the $\aleph_0$-continuous case.

Proposition 3. *If M is a nonsingular quasi-$\aleph_0$-continuous R-module, then the following are equivalent*:

a) *M is directly finite.*

b) *M contains no infinite direct sums of nonzero pairwise isomorphic submodules.*

c) *Any submodule of M is directly finite.*

An R-module M is said to have the *finite exchange property* if, for any direct decomposition $G=M'\oplus C=\oplus_{i\in I}D_i$ with $M'\cong M$ and the index set I finite, there are submodules $D_i'\leq D_i$ such that $G=M'\oplus(\oplus_{i\in I}D_i')$.

Lemma 4 ([7, Corollary 4]). *Every projective module over a regular ring has the finite exchange property.*

We are now in a position to prove the main theorem, which generalizes [2, Corollary 14.26] (cf. [6, Theorem 4] for quasi-continuous modules).

Theorem 5. *Let M be a quasi-$\aleph_0$-continuous directly finite projective module over a regular ring R, and let A, $B\in\mathcal{A}(M)$.*

(a) *If $A\cong B$ then $A^*\cong B^*$.*

(b) *If $A\lesssim B$ then $A^*\lesssim B^*$.*

Proof. (a) Put $D=A\cap B$ and let $f:A\rightarrow B$ be an isomorphism. In view of [2, Lemma 14.10 (a)], we see that $D\in\mathcal{A}(M)$, and so there exists D^* such that $D^*<\oplus A^*$ and $D^*<\oplus B^*$ (Lemma 1). Put $A^*=D^*\oplus X$ and $B^*=D^*\oplus Y$ with some X and Y. We claim the following facts:

(1) If $L\in\mathcal{A}(M)$ and $N<\oplus L$, then $N\in\mathcal{A}(M)$, and therefore both X and Y are in $\mathcal{A}(M)$.

(2) $A^*\cap B^*=D^*$ and $X\cap Y=X\cap B^*=A^*\cap Y=0$.

Proof of (1). Put $L=N\oplus K$ with some K, and let $\pi:L\rightarrow N$ be the projection map. There exists a countably generated submodule L' such that $L'\leq_e L$. Then $\pi(L')$ is countably generated and $L'\cap N\leq\pi(L')\leq N$. Hence $\pi(L')\leq_e N$.

Proof of (2). Obviously, $D^*\leq A^*\cap B^*$. On the other hand, $D=A\cap B\leq_e A^*\cap B^*$ and $A^*\cap B^*\leq D^*$ by Lemma 1. Hence $A^*\cap B^*=D^*$, and therefore $X\cap Y\leq X\cap B^*=X\cap A^*\cap B^*=X\cap D^*=0$. Similarly, $A^*\cap Y=0$.

We claim further that there exist decompositions

$$X=X_1'\oplus X_1'',\quad X_n''=X_{n+1}'\oplus X_{n+1}'',$$
$$Y=Y_1'\oplus Y_1'',\quad Y_n''=Y_{n+1}'\oplus Y_{n+1}'',$$
$$D^*=D_1'\oplus D_1'',\quad D_n''=D_{n+1}'\oplus D_{n+1}''$$

such that

$$X_n'\cong Y_n',\ D_{2n-1}'\cong X_{2n-1}'',\ D_{2n}'\cong Y_{2n}''\quad(n=1,\ 2,\ \cdots).$$

Since $f(A\cap X)$ and $f(D)$ are in $\mathcal{A}(M)$ and $f(A\cap X)\oplus f(D)\leq_e f(A)=B$, we have $B^*=(f(A\cap X))^*\oplus f(D)^*$ by Lemma 1 and (C_3). According to Lemma 4, there exist decompositions $Y=Y_1'\oplus Y_1''$ and $D^*=D_1'\oplus D_1''$

such that $Y\oplus D^*=B^*=(f(A\cap X))^*\oplus Y_1''\oplus D_1''$. Then we have an isomorphism $g: Y_1''\oplus D_1''\to (f(D))^*$. Noting that $A\cap X\leq_e X$ and $(A\cap X)\cap f(A\cap X)\leq X\cap B^*=0$, we see that $X\cong (f(A\cap X))^*\cong Y_1'\oplus D_1'$ by Lemma 2, and so there exists a decomposition $X=X_1'\oplus X_1''$ with $X_1'\cong Y_1'$ and $h: D_1'\cong X_1''$. Putting here $E=g^{-1}f(D)$ and $k=(h\oplus$ the identity map on $D_1'')f^{-1}g|_E: E\to X_1''\oplus D_1''$. Since $k(Y_1''\cap E)\oplus k(D_1''\cap E)\leq_e X_1''\oplus D_1''$, we have

$$X_1''\oplus D_1''=X_1''\oplus (D_1''\cap E)^*=(k(Y_1''\cap E))^*\oplus (k(D_1''\cap E))^*,$$

where $Y_1''\cong (k(Y_1''\cap E))^*$. Then, by the above discussion, there exist decompositions

$$X_1''=X_2'\oplus X_2'',\ Y_1''=Y_2'\oplus Y_2'' \text{ and } D_1''=(D_1''\cap E)^*=D_2'\oplus D_2''$$

with $X_1''\oplus D_1''=(k(Y_1''\cap E))^*\oplus X_2''\oplus D_2''$ and isomorphisms $g_1: X_2''\oplus D_2''\cong (k(D_1''\cap E))^*$, $X_2'\cong Y_2'$ and $h_1: D_2'\cong Y_2''$. Repeating this procedure successively, we obtain the desired decompositions.

Next, we claim that $\oplus_{n=1}^{\infty}X_n'\leq_e X$ and $\oplus_{n=1}^{\infty}Y_n'\leq_e Y$ (cf. the proof of [3, Theorem 1.4]). Suppose that $C\cap(\oplus_{n=1}^{\infty}X_n')=0$ for a cyclic submodule C of X. Obviously, $C\in\mathcal{A}(M)$, $C<\oplus X<\oplus M$ and $X_1'\oplus\cdots\oplus X_{2n-1}'\oplus X_{2n-1}''=X<\oplus M$. Since $C\oplus(X_1'\oplus\cdots\oplus X_{2n-1}')<\oplus X$ by (C_3), we see that $C\lesssim X_{2n-1}''\cong D_{2n-1}'$, and so $C\oplus C\oplus\cdots\lesssim D_1'\oplus D_3'\oplus\cdots\leq D^*<\oplus M$. Thus, $C=0$ by Proposition 3. Similarly, we can show that $\oplus_{n=1}^{\infty}Y_n'\leq_e Y$.

Now, noting that $(\oplus_{n=1}^{\infty}X_n')\cap(\oplus_{n=1}^{\infty}Y_n')\leq X\cap Y=0$ by (2) and that $\oplus_{n=1}^{\infty}X_n'\cong\oplus_{n=1}^{\infty}Y_n'\in\mathcal{A}(M)$ by (1), we see that $X\cong Y$ by Lemma 2, and so $A^*=D^*\oplus X\cong D^*\oplus Y=B^*$.

(b) If $f: A\to B$ is a monomorphism, then $A\cong f(A)\in\mathcal{A}(M)$ and $f(A)\leq B$. Hence $A^*\cong(f(A))^*\leq B^*$ by (a) and Lemma 1.

Corollary. *Let M be a quasi-$\aleph_0$-continuous directly finite projective module over a regular ring R such that $M\in\mathcal{A}(M)$. Then the following are equivalent:*

a) *M is $\aleph_0$-continuous.*

b) *Every injective endomorphism of M is an automorphism.*

An R-module M is said to have the *cancellation property* if $M\oplus H\cong M\oplus K$ implies always $H\cong K$, or equivalently if $M\oplus H=N\oplus K$ with $M\cong N$ implies $H\cong K$ (cf. [1]).

From the proof of [1, Theorem 2], we see the next

Lemma 6. *Let M be an R-module with the finite exchange property.*

Then M has the cancellation property if and only if isomorphic direct summands of M have isomorphic complements.

Theorem 7. *Let M be a quasi-$\aleph_0$-continuous directly finite projective module over a regular ring R. If $M \in \mathcal{A}(M)$, then M has the cancellation property.*

Proof. Let $M=A\oplus C=B\oplus D$ with $C\cong D$ for some R-modules A, B, C and D. Since $M\in\mathcal{A}(M)$, A, B, C and D are also in $\mathcal{A}(M)$ by the statement (1) in the proof of Theorem 5. Then $A\cong B$ by Proposition 3, Lemma 4, Theorem 5 and the proof of [3, Lemmas 1.2, 1.3 and Theorem 1.4]. Hence, by Lemmas 4 and 6, M has the cancellation property.

Theorem 8. *Let R be a regular ring. If M and N are quasi-$\aleph_0$-continuous directly finite projective R-modules, then $M\oplus N$ is directly finite.*

Proof. Let $M\oplus N=X\oplus Y$ for some submodules X and Y with an isomorphism $f: M\oplus N\to X$. Take a cyclic submodule yR of Y. Then $\{yR, f(yR), f^2(yR), \cdots\}$ is an infinite independent sequence of pairwise isomorphic cyclic submodules of $M\oplus N$. Noting that $yR <\oplus M\oplus N$, we have decompositions $M=M_1'\oplus M_1''$ and $N=N_1'\oplus N_1''$ such that $M\oplus N=yR\oplus M_1''\oplus N_1''$ and $yR\cong M_1'\oplus N_1'$ (Lemma 4). Next, noting that $yR\oplus f(yR) <\oplus yR\oplus M_1''\oplus N_1''$, again by Lemma 4 we have decompositions $M_1''=M_2'\oplus M_2''$ and $N_1''=N_2'\oplus N_2''$ such that $M\oplus N=yR\oplus f(yR)\oplus M_2''\oplus N_2''$ and $f(yR)\cong M_2'\oplus N_2'$. Continuing this procedure, we have decompositions

$$M=M_1'\oplus M_1'',\ M_n''=M_{n+1}'\oplus M_{n+1}'',$$
$$N=N_1'\oplus N_1'',\ N_n''=N_{n+1}'\oplus N_{n+1}''$$

such that $yR\cong M_1'\oplus N_1'$ and $f^n(yR)\cong M_{n+1}'\oplus N_{n+1}'$ $(n=1, 2, \cdots)$, where M_n' and N_n' are cyclic R-modules.

Put $A=M_1'\oplus M_2'\oplus\cdots$, $B=N_1'\oplus N_2'\oplus\cdots$, $C=M_2'\oplus M_3'\oplus\cdots$, and $D=N_2'\oplus N_3'\oplus\cdots$. Then $A=M_1'\oplus C$, $B=N_1'\oplus D$, and there exists an isomorphism $g: A\oplus B\to C\oplus D$. Since $g(A)$ is projective, there exist decompositions $C=C'\oplus C''$ and $D=D'\oplus D''$ such that $C\oplus D=g(A)\oplus C''\oplus D''$ (Lemma 4). Then $A\cong g(A)\cong C'\oplus D'$ and $B\cong g(B)\cong C''\oplus D''$, and so there exist decompositions $A=A'\oplus A''$ and $B=B'\oplus B''$ with $A'\cong C'$, $m: A''\cong D'$, $l: B'\cong C''$ and $B''\cong D''$. Noting that $A'\oplus A''=M_1'\oplus C'\oplus C''$ and $A'\cong C'$, by Theorem 5 (a) we see that $A'^*\oplus A''^*=M_1'\oplus C'^*\oplus C''^*$ and $A'^*\cong C'^*$ in M. Since A'^* is a quasi-$\aleph_0$-continuous directly finite projective module belonging to $\mathcal{A}(A'^*)$, we have an isomorphism $h: A''^*\cong$

$M_1' \oplus C''^*$ by Theorem 7. Similarly, we get an isomorphism $k: B'^* \cong N_1' \oplus D'^*$. Now, we put $E = l(k^{-1}(k(B') \cap D'))$. Then $m^{-1}(k(B') \cap D') \cong E \leq C''$ and $m^{-1}(k(B') \cap D') (\in \mathcal{A}(M))$ is an essential submodule of A'', and so $A''^* \cong E^* <\oplus C''^*$ in M by Theorem 5 (a), and $A''^* = h^{-1}(M_1') \oplus h^{-1}(C''^*) \oplus > h^{-1}(M_1') \oplus h^{-1}(E^*)$. Since A''^* is directly finite, we have $h^{-1}(M_1') = 0$, and so $M_1' = 0$; likewise $N_1' = 0$ Hence $yR \cong M_1' \oplus N_1' = 0$, and so $Y = 0$, which means that $M \oplus N$ is directly finite.

A regular ring R is said to satisfy the *comparability axiom* provided that, for any $x, y \in R$, either $xR \lesssim yR$ or $yR \lesssim xR$ ([2, p. 80]). It is known that if R is a regular ring satisfying the comparability axiom then, for any finitely generated projective R-modules A and B, either $A \lesssim B$ or $B \lesssim A$ ([2, Proposition 8.2]).

Theorem 9. *Let R be a regular ring satisfying the comparability axiom. Then every quasi-$\aleph_0$-continuous directly finite projective R-module P is finitely generated, and hence $\aleph_0$-continuous.*

Proof. Suppose, to the contrary, that P is not finitely generated. Then, by [4], $P = \oplus_{i \in I} x_i R$ for some infinite index set I. Given $j \in I$, we set $N_j = \{i \in I \mid x_j R \lesssim x_i R\}$. Then, N_j is a finite set by Proposition 3. We can therefore find an infinite sequence $\{i_1, i_2, \cdots\}$ in I and (non-isomorphic) monomorphisms $f_{i_{n+1}}: x_{i_{n+1}} R \to x_{i_n} R (n = 1, 2, \cdots)$. Then there exist nonzero $a_{i_n} R$ such that $x_{i_n} R = f_{i_{n+1}}(x_{i_{n+1}} R) \oplus a_{i_n} R$. Since $\oplus_{n=1}^{\infty} x_{i_n} R = (\oplus_{n=1}^{\infty} f_{i_{n+1}}(x_{i_{n+1}} R)) \oplus (\oplus_{n=1}^{\infty} a_{i_n} R)$ and $\oplus_{n=2}^{\infty} x_{i_n} R$ are quasi-$\aleph_0$-continuous directly finite countably generated projective R-modules, Theorem 7 shows that $x_{i_1} R \cong \oplus_{n=1}^{\infty} a_{i_n} R$, which is a contradiction.

A regular ring is called *abelian* if all idempotents in R are central ([2, p, 25]). As the following examples show, the assumtion "the comparability axiom" cannot be deleted in Theorem 9.

Example 1. Let R be an abelian right $\aleph_0$-continuous regular ring. Then, it is easy to see that every right ideal of R is an $\aleph_0$-continuous directly finite R-module.

Example 2. Let I be an index set. For each $i \in I$, consider a ring R_i and an R_i-module M_i. Put $R = \prod_{i \in I} R_i$ and $M = \oplus_{i \in I} M_i$.

(a) If every R_i-module M_i is quasi-$\aleph_0$-continuous (resp. $\aleph_0$-continuous), then so is the R-module M.

(b) If every R_i-module M_i is directly finite, then so is the R-module M.

(c) In particular, if every R_i is a right $\aleph_0$-continuous directly finite regular ring, then the right R-module R is $\aleph_0$-continuous and directly finite.

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