Math. J. Okayama Univ. 25 (1983), 153—156

NOTE ON THE MAXIMAL RIGHT $\aleph_0$-QUOTIENT RING OF A REGULAR RING

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Throughout, R will represent a ring with unity, and $\mathfrak{A}(R)$ (resp. $\mathfrak{B}(R)$) the set of all countably generated right ideals (resp. all countably generated, essential right ideals) of R. If a right R-module M is essential in N, we write $M \subset_e N$.

The present objective is to prove that the maximal right $\aleph_0$-quotient ring of a regular ring R is a right $\aleph_0$-continuous, regular ring if and only if R satisfies the $\aleph_0$-complement property, i.e., every countably generated right ideal has a countably generated complement (Theorem 4). We also show that the $\aleph_0$-complement property is Morita invariant. As for terminologies and fundamental results used in this note, we refer to Goodearl [1].

We begin with the following

Lemma 1 ([1, Prop. 14.11]). *Let R be a regular ring, and Q the maximal right quotient ring of R. Let $S=\{x \in Q : xJ \subset R$ for some $J \in \mathfrak{B}(R)\}$. Then there holds the following:*

(1) *S is a subring of Q, and is the maximal right $\aleph_0$-quotient ring of R.*

(2) *For any $K \in \mathfrak{B}(R)$, every map in $\mathrm{Hom}_S(K, S_S)$ extends to S.*

We say that R satisfies the *right $\aleph_0$-complement property* if for any $I \in \mathfrak{A}(R)$ there exists some $J \in \mathfrak{A}(R)$ such that $I \cap J=0$ and $I \oplus J \in \mathfrak{B}(R)$. Obviously, every regular ring having no uncountable direct sums of non-zero right ideals satisfies the $\aleph_0$-complement property, and also every right $\aleph_0$-continuous regular ring satisfies the same ([1, Lemma 14.8]). We shall give an example of a regular ring satisfying the $\aleph_0$-complement property which is not right $\aleph_0$-continuous and has an uncountable direct sum of non-zero right ideals.

Example (cf. [1, Example 13.8]). Let F be a field having a proper subfield K. Let I be an uncountable well-ordered set, and set $F_\alpha=F$, $K_\alpha=K$ $(\alpha \in I)$. Let $Q=\prod_{\alpha \in I} F_\alpha$ and $R=\{x \in Q : x_\alpha \in K_\alpha$ for all but finitely many $\alpha\}$. Observing that R contains all the idempotents of Q, we see that R is a continuous regular ring. Now, by Proposition 5 below, $T=M_2(R)$ satisfies the $\aleph_0$-complement property. We identify T with the subring $\{A\in\prod_{\alpha\in I}M_2(F_\alpha): A_\alpha\in M_2(K_\alpha)$ for all but finitely many $\alpha\}$. Evidently, T has an uncountable direct sum of non-zero right ideals. Since the lattice of principal right ideals of T is not $\aleph_0$-complete by [1, Example 13.8], T is not right $\aleph_0$-continuous.

Lemma 2. *Let R be a regular ring satisfying the right $\aleph_0$-complement property, and S the maximal right $\aleph_0$-quotient ring of R. Then, for any $I\in\mathfrak{A}(R)$ and any R-homomorphism $f: I\longrightarrow R$, there exists an element s in S such that $f(a)=sa$ for all $a\in I$.*

Proof. Choose $J\in\mathfrak{A}(R)$ with $I\oplus J\subset_e R$, and extend f to a homomorphism $I\oplus J\longrightarrow R$ by setting $f(J)=0$. As is easily seen, $IS\oplus JS\subset_e S_S$. Then, noting that $K\otimes_R S\cong KS$ for any right ideal K of R ([1, Cor. 1.13]), we can define an S-homomorphism $f^*: IS\oplus JS\longrightarrow S$ by setting $f^*(\sum_i a_ix_i+\sum_j b_jy_j)=\sum_i f(a_i)x_i$ ($a_i\in I$, $b_j\in J$, x_i, $y_j\in S$). Since $IS\oplus JS\subset_e S_S$, [1, Th. 14.11] shows that there exists $s\in S$ such that $f^*(x)=sx$ for all $x\in IS\oplus IS$, completing the proof.

Lemma 3. *If a regular ring R satisfies the right $\aleph_0$-complement property, then so does the maximal right $\aleph_0$-quotient ring S of R, and conversely.*

Proof. Let $K=\sum_n x_nS\in\mathfrak{A}(S)$. By definition, there exists $J_n\in\mathfrak{B}(R)$ such that $x_nJ_n\subset R$ ($n=1, 2, \cdots$). Then, $I=\sum_n x_nJ_n\in\mathfrak{A}(R)$. Consider the natural epimorphism $\bigoplus_n S/J_n\longrightarrow K/I$. Inasmuch as $J_n\subset_e R\subset_e S_R$ for all n, we see that $(K/I)_R$ is singular, and therefore $I\subset_e K_R$. Then we have $I\subset_e K_S$. Choose $J\in\mathfrak{A}(R)$ such that $I\oplus J\subset_e R_R$. Then we have $IS\oplus JS\subset_e S_S$, and hence $K\oplus JS\subset_e S_S$.

Conversely, suppose that S has the right $\aleph_0$-complement property. Given $C\in\mathfrak{A}(R)$, there exists $K=\sum_n x_nS\in\mathfrak{A}(S)$ such that $CS\oplus K\in\mathfrak{B}(S)$. By definition, we then have $CS\oplus K\subset_e S_R$. By making use of the above argument, we can find $I\in\mathfrak{A}(R)$ such that $I\subset_e K_R$. Now, combining this with $C\subset_e CS_R$, we get $C\oplus I\subset_e S_R$, and therefore $C\oplus I\in\mathfrak{B}(R)$.

We are now in a position to state our main theorem.

Theorem 4. *Let R be a regular ring, and S the maximal right $\aleph_0$-quo-*

tient ring of R. Then the following conditions are equivalent :

1) *R satisfies the right $\aleph_0$-complement property.*

2) *S is a right $\aleph_0$-continuous regular ring.*

Proof. 1)$\Rightarrow$2). In view of Lemma 2, we can prove that S is regular, in the same way as in the proof of [1, Th. 14.12]. Now, let $K \in \mathfrak{A}(S)$. Then $K \oplus J \subset_e S_S$ with some $J \in \mathfrak{A}(S)$. According to Lemma 1 (2), the natural projection map $K \oplus J \longrightarrow K$ is given by the left multiplication of an element p in S. Obviously, p is an idempotent and $K \subset pS$. Since $(pS/K)_S$ is singular as the homomorphic image of the singular module $(S/(K \oplus J))_S$, we see that $K \subset_e pS_S$, and hence S is right $\aleph_0$-continuous by [1, Cor. 14.4].

2)$\Rightarrow$1). Obviously, S satisfies the right $\aleph_0$-complement property. Hence, by Lemma 3, R satisfies the same property.

In general, the $\aleph_0$-continuity of a regular ring is not Morita invariant. However, we shall show that the $\aleph_0$-complement property of a regular ring is Morita invariant. We can prove the next in the same way as in the proof of [1, Lemma 14.18].

Lemma 5. *Let R be a regular ring with $\aleph_0$-complement property, and P a finitely generated projective right R-module. If A is a countably generated submodule of P, then $A \oplus B \subset_e P$ with some countably generated submodule B of P.*

Proposition 6. *Let P be a finitely generated projective right module over a regular ring R. If R satisfies the $\aleph_0$-complement property, then* $\mathrm{End}_R(P)$ *does the same property.*

Proof. Put $T = \mathrm{End}_R(P)$, and $M \in \mathfrak{A}(T)$. Then, by [1, Prop. 2.14], there exist orthogonal idempotents $f_1, f_2, \cdots$ in T such that $M = \bigoplus_n f_n T$. We put $A = \bigoplus_n f_n(P)$, and choose a countably generated submodule B of P such that $A \oplus B \subset_e P$ (Lemma 5). In view of [1, Prop. 2.13], there exist countable idempotents $g_1, g_2, \cdots$ in T such that $g_{n+1}g_n = g_n$ $(n=1, 2, \cdots)$ and $B = \bigcup_n g_n(P)$. Put $N = \bigcup_n g_n T \in \mathfrak{A}(T)$. It is easy to see that $M \cap N = 0$. Now, we shall show that $M \oplus N \subset_e T_T$. Let e be an arbitrary non-zero idempotent of T. Since $A \oplus B \subset_e P$ implies $(A \oplus B) \cap e(P) \neq 0$, we can find $x \neq 0$ in $(A \oplus B) \cap e(P)$. If α is a projection of P on the direct summand xR (see [1, Th. 1.11]), we can easily see that $\alpha \in (M+N) \cap eT$. We have thus completed the proof.

REFERENCE

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(*Received, June 30, 1983*)