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SOME REMARKS ON NORMAL CLASSES OF SEMIPRIME RINGS

MOTOSHI HONGAN

The purpose of this note is to extend the results of [4] and [5], which are obtained for normal classes of prime rings, to those of semiprime rings. As for notations and terminologies used in this paper we follow the previous paper [3].

We begin with the following

Proposition 1 (cf. [4, Proposition 3]). *Let (R,V,W,S) be a Morita context with $R\neq 0$, and write $C=\begin{pmatrix} R & V \\ W & S\end{pmatrix}$. Then C is a semiprime ring if and only if the following hold:*

1) *R is a semiprime ring.*
2) *$Vw=0$ $(w\in W)$ implies $w=0$.*
3) *$vW=0$ $(v\in V)$ implies $v=0$.*
4) *$S=0$ or S is a semiprime ring.*

Proof. Observe that the lack of symmetry in 2) and 3) is only apparent. For example, $wV=0$ implies $(Vw)^2=0$, so $w=0$ by 1) and 2). Similarly, $Wv=0$ implies $v=0$. To see that C is a semiprime ring, suppose $cCc=0$, where $c=\begin{pmatrix} r & v \\ w & s\end{pmatrix}\in C$. Since $0=\begin{pmatrix} r & v \\ w & s\end{pmatrix}\begin{pmatrix} R & 0 \\ 0 & 0\end{pmatrix}\begin{pmatrix} r & v \\ w & s\end{pmatrix}=\begin{pmatrix} rRr & rRv \\ wRr & wRv\end{pmatrix}$ and R is semiprime, we have $r=0$, and then $0=\begin{pmatrix} 0 & v \\ w & s\end{pmatrix}\begin{pmatrix} 0 & 0 \\ W & 0\end{pmatrix}\begin{pmatrix} 0 & v \\ w & s\end{pmatrix}=\begin{pmatrix} 0 & vWv \\ 0 & sWv\end{pmatrix}$, whence it follows $vWvW=0$. Now, by the semiprimeness of R, we have $vW=0$ and $v=0$. Similarly, we can obtain $w=0$. Hence, $0=\begin{pmatrix} 0 & 0 \\ 0 & s\end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & S\end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & s\end{pmatrix}=\begin{pmatrix} 0 & 0 \\ 0 & sSs\end{pmatrix}$, and so $s=0$ by 4). The converse is easily checked.

Let (R,V,W,S) be a Morita context, and A an ideal of R. We set $V_A=\{v\in V \mid vW\subseteq A\}$, $W_A=\{w\in W \mid Vw\subseteq A\}$ and $S_A=\{s\in S \mid VsW\subseteq A\}$. Then it is known that $(R/A, V/V_A, W/W_A, S/S_A)$ is a Morita context, the products being defined in the natural manner. If X is a subset of a ring R, we denote by $\mathrm{Ann}_R(X)=l_R(X)\cap r_R(X)$ the annihilator of X in R. In case L is a left, right or two-sided ideal of R, we write $L\lhd_l R$, $L\lhd_r R$ or $L\lhd R$, respectively.

Now, we shall extend [4, Theorem 1] to the classes of semiprime rings.

Theorem 1. *Let $\mathcal{P}$ be a class of semiprime rings. Then $\mathcal{P}$ is a normal class if and only if $\mathcal{P}$ satisfies the following conditions*:

(i) *If $R\in\mathcal{P}$, $L\triangleleft_l T\triangleleft_r R$ and L is a semiprime ring, then $L\in\mathcal{P}$.*

(ii) *If R is a semiprime ring, $L\triangleleft_l T\triangleleft_r R$, $\mathrm{Ann}_T(L)=\mathrm{Ann}_R(T)=0$ and $L\in\mathcal{P}$, then $R\in\mathcal{P}$.*

Proof. Suppose that $\mathcal{P}$ satisfies (i) and (ii). Let (R,V,W,S) be an S-faithful Morita context with $R\in\mathcal{P}$. Then we have a Morita context $(R,V/V_{(0)},W/W_{(0)},S)$, which satisfies the conditions 1)–3) in Proposition 1. Suppose that $sSs=0$ $(s\in S)$. Then $sWVs=0$ implies $(VsW)^2=0$, and so $VsW=0$. This means $s=0$, that is, S is a semiprime ring, proving 4) in Proposition 1. Hence the ring $C=\begin{pmatrix} R & V/V_{(0)} \\ W/W_{(0)} & S\end{pmatrix}$ is semiprime by Proposition 1. If we set $R'=\begin{pmatrix} R & 0 \\ 0 & 0\end{pmatrix}$ and $T=\begin{pmatrix} R & V/V_{(0)} \\ 0 & 0\end{pmatrix}$, then $R\simeq R'\triangleleft_l T\triangleleft_r C$. As is easily seen, $\mathrm{Ann}_T(R')=0$ and $\mathrm{Ann}_C(T)=0$, and so $C\in\mathcal{P}$ by (ii). Again, $S\simeq\begin{pmatrix} 0 & 0 \\ 0 & S\end{pmatrix}\triangleleft_l\begin{pmatrix} 0 & 0 \\ W/W_{(0)} & S\end{pmatrix}\triangleleft_r C$ and S is a semiprime ring, and so (i) implies $S\in\mathcal{P}$.

Conversely, suppose that $\mathcal{P}$ is a normal class. If $R\in\mathcal{P}$, $L\triangleleft_l T\triangleleft_r R$ and L is a semiprime ring, then the context (R,RL,T,L) is L-faithful, and so $L\in\mathcal{P}$. If R is a semiprime ring, $L\triangleleft_l T\triangleleft_r R$, $\mathrm{Ann}_T(L)=\mathrm{Ann}_R(T)=0$ and $L\in\mathcal{P}$, then the context (L,T,RL,R) is R-faithful, and so $R\in\mathcal{P}$.

Corollary 1 ([3, Theorem 3.2]). *Every normal class $\mathcal{P}$ of semiprime rings is a weakly special class.*

Now, combining Proposition 1 and the proof of Theorem 1, we readily obtain

Corollary 2 (cf. [4, Corollary 2 to Theorem 1]). *Let $\mathcal{P}$ be a normal class of semiprime rings. Let (R,V,W,S) be a Morita context with $R\in\mathcal{P}$, and $C=\begin{pmatrix} R & V \\ W & S\end{pmatrix}$. If C is a semiprime ring, then C is in $\mathcal{P}$.*

Now, we expose the relationship between the normal classes of semiprime rings and the weakly special classes.

Theorem 2 (cf. [5, Theorem 7.5]). *Let $\mathcal{P}$ be a class of semiprime rings. Then $\mathcal{P}$ is a normal class if and only if $\mathcal{P}$ satisfies the following conditions:*

(i) *$\mathcal{P}$ is a weakly special class.*

(ii) *If $R \in \mathcal{P}$ then $eRe \in \mathcal{P}$ for every non-zero idempotent e of R.*

(iii) *If e is a non-zero idempotent of a semiprime ring R and $eRe \in \mathcal{P}$, then $R/\mathrm{Ann}_R(ReR) \in \mathcal{P}$.*

Proof. If $\mathcal{P}$ is a normal class, then (i), (ii) and (iii) are satisfied (Corollary 1 and [3, Proposition 3.2]).

Conversely, suppose that $\mathcal{P}$ is a weakly special class with the properties (ii) and (iii). Let (R,V,W,S) be an S-faithful Morita context with $R \in \mathcal{P}$. If R^1 is the Dorroh extension of R obtained by adjoining identity in the usual way, then the context (R^1,V,W,S) is S-faithful. Let $A = l_{R^1}(R)$. Then A is an ideal of R^1 and, in the notations introduced just before Theorem 1, we have $V_A = V_{(0)}$, $W_A = W_{(0)}$ and $S_A = S_{(0)} = 0$. If we set $R^\circ = R/A$, $\bar{V} = V/V_A$ and $\bar{W} = W/W_A$, then R° is a ring with an identity and is contained in $\mathcal{P}$ by [2, Theorems 1 and 5]. And $(R^\circ,\bar{V},\bar{W},S)$ is an S-faithful Morita context. Now, we set $C = \begin{pmatrix} R^\circ & \bar{V} \\ \bar{W} & S \end{pmatrix}$ and $e_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Then $e_{11}Ce_{11} \simeq R^\circ \in \mathcal{P}$, and hence $C/\mathrm{Ann}_C(Ce_{11}C) \in \mathcal{P}$ by (iii). But, we can easily see that $\mathrm{Ann}_C(Ce_{11}C) = 0$, and so $C \in \mathcal{P}$. Next, we consider the Dorroh extension S^1 of S and $C^1 = \begin{pmatrix} R^\circ & \bar{V} \\ \bar{W} & S^1 \end{pmatrix}$. Then C^1 is a ring and $l_{C^1}(C) = \begin{pmatrix} 0 & 0 \\ 0 & l_{S^1}(S) \end{pmatrix}$. Setting $S^\circ = S^1/l_{S^1}(S)$ and $C^\circ = C^1/l_{C^1}(C)$, we see that $C^\circ = \begin{pmatrix} R^\circ & \bar{V} \\ \bar{W} & S^\circ \end{pmatrix}$ is a ring with an identity containing C as an ideal. Now, by checking the conditions 2)–4) in Proposition 1, we shall prove that C° is a semiprime ring. First, if $\bar{V}\bar{w} = 0$ ($\bar{w} \in \bar{W}$), namely $Vw \subseteq l_{R^1}(R)$, then $VwR = 0$, and so $Vw = 0$, that is, $w \in W_{(0)}$. Similarly, $\bar{v}\bar{W} = 0$ ($\bar{v} \in \bar{V}$) implies $\bar{v} = 0$. Finally, if $\overline{(s,n)}S^\circ\overline{(s,n)} = 0$ ($\overline{(s,n)} \in S^\circ$), then $(s,n)S(s,n)S = 0$. Since C is semiprime, S is so. Hence $(s,n)S = 0$, i.e., $\overline{(s,n)} = 0$, proving that S° is a semiprime ring. Thus, we have shown that C° is a semiprime ring. Since C is an ideal of C° and C is in the weakly special class $\mathcal{P}$, we have $C^\circ/\mathrm{Ann}_{C^\circ}(C) \in \mathcal{P}$. As is easily verified, $l_{C^\circ}(C) = 0$, and so $C^\circ \in \mathcal{P}$. Then $S^\circ \simeq e_{22}C^\circ e_{22} \in \mathcal{P}$ by (ii), where $e_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \in C^\circ$. Since S is an ideal of S° and $\mathcal{P}$ is a weakly special class, we obtain $S \in \mathcal{P}$. Thus, $\mathcal{P}$ is a normal class.

Given subsets X and Y of a ring R, we set $YX^{-1} = \{a \in R \mid aX \subseteq Y\}$ and $X^{-1}Y = \{a \in R \mid Xa \subseteq Y\}$.

The next result extends [5, Theorem 7.6] to the classes of semiprime rings.

Theorem 3. *Let $\mathcal{P}$ be a normal class of semiprime rings, and (R, V, W, S) an S-faithful Morita context. Then there is a one-to-one correspondence between* $\{A \lhd R \mid R/A \in \mathcal{P},\ A(VW)^{-1}(=(VW)^{-1}A) \subseteq A \text{ and } A \not\supseteq VW\}$ *and* $\{B \lhd S \mid S/B \in \mathcal{P},\ B(WV)^{-1}(=(WV)^{-1}B) \subseteq B \text{ and } B \not\supseteq WV\}$.

Proof. Let A be an ideal of R such that $R/A \in \mathcal{P}$, $A(VW)^{-1} \subseteq A$ and $A \not\supseteq VW$. Here we set $\bar{R}=R/A$, $\bar{V}=V/V_A$, $\bar{W}=W/W_A$ and $\bar{S}=S/S_A$. Now, suppose that $\bar{S}=0$, then $VSW \subseteq A$ which implies $VWVW \subseteq VSW \subseteq A$, so $VW \subseteq A$ by the semiprimeness of A, a contradiction. Hence $(\bar{R}, \bar{V}, \bar{W}, \bar{S})$ is an $\bar{S}$-faithful Morita context, and so $\bar{R} \in \mathcal{P}$ implies $\bar{S}=S/S_A \in \mathcal{P}$. Assume now that $WV \subseteq S_A$, then $VWVW \subseteq A$, and so $VW \subseteq A$, a contradiction. Hence we have $WV \subseteq S_A$. If x is any element in $S_A(WV)^{-1}$, then $xWV \subseteq S_A$, i.e., $VxWVW \subseteq A$, and so $(VxW)^2 \subseteq A$. Hence we have $VxW \subseteq A$, and so $x \in S_A$, proving that $S_A(WV)^{-1} \subseteq S_A$. Now, let $R_{S_A}=\{r \in R \mid WrV \subseteq S_A\}$. Since $V(WaV)W=(VW)a(VW) \subseteq A$ for any $a \in A$, we have $R_{S_A} \supseteq A$. If $r \in R_{S_A}$, then $WrV \subseteq S_A$, that is, $VWrVW \subseteq A$. Since A is semiprime, this means $rVW \subseteq A$, i.e., $r \in A(VW)^{-1} \subseteq A$. Hence we have $R_{S_A} \subseteq A$, and therefore $R_{S_A}=A$. By symmetry, we can get the inverse map out of B.

Proposition 2 (cf. [5, Corollary 7.8]). *Let R, S be semiprime rings with a common non-zero ideal A such that $l_R(A)=l_S(A)=0$.*

(1) *Let $\mathcal{P}$ be a normal class of semiprime rings. Then, R is in $\mathcal{P}$ if and only if so is S.*

(2) *R is a semiprimitive ring if and only if so is S.*

(3) *R is a right Goldie ring if and only if so is S.*

Proof. (1) Consider the S-faithful Morita context (R, A, A, S) and the R-faithful Morita context (S, A, A, R).

(2) Since the class of semiprimitive rings is normal by [1, Corollary 21], this is immediate by (1).

(3) By assumption, A is an essential right ideal of R. If R is a right Goldie ring, then A contains a regular element a of R. Then, for any q in the classical right quotient ring $Q(R)$ of R, $a^{-1}qa=bc^{-1}$ with some $b, c \in R$, and therefore $q=aba(aca)^{-1}$. Hence, $Q(A)=Q(R)$, and A is a semiprime right Goldie ring (by Goldie's theorem). Conversely, if A is a semiprime right Goldie ring, then $A \hookrightarrow R \hookrightarrow \mathrm{End}(A_A) \hookrightarrow Q(A)$, and so $Q(A)=Q(R)$. Thus, R is a right Goldie ring. Similarly, we can show that S is a right

Goldie ring if and only if so is A. This completes the proof.

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TSUYAMA COLLEGE OF TECHNOLOGY

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