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SOME REMARKS ON THE SET OF IDEMPOTENTS AND THE SET OF ELEMENTS SQUARE ZERO

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Throughout the present paper, R will represent an (associative) ring different from 0. Given a subset S of R, we denote by $\langle S\rangle$ the subring of R generated by S. Let E be the set of idempotents in R, and N the set of nilpotents in R. We set $N^*=\{x\in R \mid x^2=0\}$. We say R is *normal* if and only if E is contained in the center of $R:[E,R]=0$. As is well known, if $[E,E]=0$ or $[E,N^*]=0$ then R is normal. But, $[N^*,N^*]=0$ need not yield the normality of R. If every left (resp. right) ideal of R is idempotent, R is said to be *fully left* (resp. *right*) *idempotent*. Finally, R is called a *chain ring* if the lattice of ideals of R forms a chain.

In §1, we investigate the structure of π-regular rings with only finitely many idempotents (Theorem 1), which leads us to the main theorem of Putcha and Yaqub [7]. Also, we give the conditions for an algebra generated by idempotents over a commutative regular ring to be commutative (Theorem 2). In §2, our interest will be directed towards to rings whose elements square zero commute with each other. First, such a semiprime P.I. ring is shown to be reduced (Theorem 3). Next, by making use of a generalization of the corrected statement of [8, Proposition 4], we prove that such a fully left and right idempotent chain ring is a simple ring without nontrivial idempotents (Theorem 4).

1. We state first the following

Theorem 1. *Let R be a ring in which every non-nil right ideal contains a nonzero idempotent. If E is finite then either R is a nil ring or the residue class ring $\bar{R}$ of R modulo its (Jacobson) radical J is a finite direct sum of finite simple rings and/or division rings.*

Proof. If $R=J$ then R is obviously a nil ring. If $R\neq J$ then, in view of [4, Theorem 2.1], we see that $\bar{R}$ is Artinian (semiprimitive) and, needless to say, every simple constituent of $\bar{R}$ is either finite or a division ring.

Corollary 1 ([7, Theorem 1]). *If $R\backslash N$ is finite, then R is either finite or nil (and conversely).*

Proof. Obviously, R is strongly π-regular. Hence the radical J of R is a nil ideal, and $\bar{R}=R/J$ also has only a finite number of nonnilpotents. Suppose $\bar{R}$ is nonzero. Then it is easy to see that J is finite. On the other hand, by Theorem 1, $\bar{R}$ is a finite semiprimitive ring, and therefore R is finite.

Remark 1. On contrast with Corollary 1, in an unpublished paper, M.S. Putcha and A. Yaqub proved that if $R\backslash E$ is finite then R is either finite or Boolean (and conversely).

In advance of stating the second corollary to Theorem 1, we prove the next, which concerns [10, Problem 94].

Proposition 1. *The following statements are equivalent*:

1) *R has a finite subset F such that every nonzero right ideal of R has a nonzero* (*nonempty*) *intersection with F.*

2) *The right socle S of R is essential in R_R and the lattice $\mathcal{L}(S_R)$ of its submodules is finite. When this is the case, $\bar{S}=(S+P)/P$ is a finite direct sum of finite simple rings and/or division rings, where P is the prime radical of R.*

Proof. It suffices to show that 1) implies 2). We may assume that F does not contain 0. For any $f\in F$, we denote by M_f the least right ideal of R containing f. After suitable arrangement of F, we may further assume that $M_f\cap M_{f'}=0$ provided $f\neq f'$. Then it is easy to see that M_f $(f\in F)$ are the minimal right ideals of R.

Let $\bar{R}=R/P$. Since $\bar{S}_{\bar{R}}$ is finitely generated, the ideal $\bar{S}$ is a direct summand of $\bar{R}_{\bar{R}}$, and therefore $\bar{S}$ is a ringtheoretic direct summand of the semiprime ring $\bar{R}$. This proves that $\bar{S}$ is an Artinian semiprimitive ring and $\mathcal{L}(\bar{S}_{\bar{S}})$ is finite. Consider now the $n\times n$ full matrix ring $T=\sum_{i,j=1}^{n} De_{ij}$ over a division ring D $(n>1)$. As is easily seen, the different idempotents $e_{11}+de_{21}$ $(d\in D)$ generate different right ideals of T. This enables us to see that $\bar{S}$ is a finite direct sum of finite simple rings and/or division rings.

Combining Proposition 1 with Theorem 1, we readily obtain

Corollary 2. *The following are equivalent*:

1) *R is a semiprime ring and contains a finite subset F such that every nonzero right ideal of R has a nonzero intersection with F.*

2) *R is a semiprimitive π-regular ring and E is finite.*

3) *R is a finite direct sum of finite simple rings and/or division rings.*

Next, we consider the commutativity conditions for certain algebras generated by its idempotents.

Proposition 2. *If $R=\langle E\rangle$ and is commutative, then R is a subdirect sum of R_α ($\alpha \in A$), where $R_\alpha = \boldsymbol{Z}$ or $\boldsymbol{Z}/p_\alpha^{k_\alpha}\boldsymbol{Z}$ with some prime p_α and $k_\alpha > 0$.*

Proof. We may, and shall, assume that R is subdirectly irreducible. Then, it is easy to see that R has 1 as the unique nonzero idempotent. If $N=0$, then we see that R is either $\boldsymbol{Z}$ or $\boldsymbol{Z}/p\boldsymbol{Z}$ with some prime p. On the other hand, if $N\neq 0$, then we can easily see that the additive order of 1 is p^k for some prime p and $k>1$. Hence $R=\boldsymbol{Z}/p^k\boldsymbol{Z}$.

Theorem 2. *Let K be a commutative regular ring with 1. If a K-algebra R is generated by E over K, then the following are equivalent :*

1) *R is commutative.*
2) *R is a subdirect sum of fields homomorphic to K.*
3) *R is a reduced ring.*
4) *R is a strongly regular ring.*

Proof. Obviously, 4) implies 3) and 2) does 3). Furthermore, as every idempotent in a reduced ring is central, 3) implies 1).

1) implies 2) and 4). Since every ideal of R is necessarily an algebra ideal, a slight modification of the proof of Proposition 2 proves 2), and so R is a reduced ring. Now, let M be a proper prime ideal of R. Then $\bar{R}=R/M$ is a domain generated by $\bar{E}$ over K. Hence, $\bar{R}$ has the identity ε as the unique nonzero idempotent, and $\bar{R}=K\cdot\varepsilon$ is a field. Thus, M is a maximal ideal, and therefore R is a regular ring (see, e.g. [1, Theorem]).

2 . We begin this section with the following

Theorem 3. *Let R be a semiprime P.I. ring. If $[N^*,N^*]=0$ then R is a reduced ring.*

Proof. By [5, Theorem 5], R has a maximal left and right quotient ring Q which is a regular ring. It suffices to show that Q is normal. Let e be an arbitrary idempotent in Q. Then there exists an essential ideal A of R such that $eA(1-e)\subseteq R$ and $(1-e)Ae\subseteq R$ (see [5, Theorem 4]). By hypothesis, $eA(1-e)Ae=(1-e)AeA(1-e)=0$. Since R is semiprime, we then have $eA(1-e)=0=(1-e)Ae$. This means that $[e,A]=0$. Now, let x be an arbitrary element of Q, and choose an essential ideal B of R such that $xB \subseteq R$. We claim that $[e,x]BA=0$. In fact, if $a \in A$ and $b \in B$ then $exba=xbae=xeba$. Hence, $[e,x]=0$, proving the normality of Q.

The next corrects and generalizes [8, Proposition 4].

Proposition 3. (1) *Let R be a fully left idempotent chain ring. If $I \neq R$ is an ideal of R then $I=(I \cap N^*)R$.*

(2) *Let R be a fully left and right idempotent chain ring. If $I \neq R$ is an ideal of R then $I=\langle I \cap N^* \rangle$.*

Proof. (1) Let x be an arbitrary element of I. Then there exists $e \in RxR$ such that $x=ex$. Furthermore, we can find $f \in ReR \subseteq RxR$ such that $fe=e$. Since $f \in RfR \subseteq R(1-f)R$ by hypothesis, we can write $e=\sum_i r_i(1-f)s_i$ with some r_i, $s_i \in R$. Hence, we get

$$x=e^2x=\sum_i er_i(1-f)s_ix \in (I \cap N^*)R.$$

(2) Let x be an arbitrary element of I. Then there exists $e \in RxR$ such that $x=ex=xe$ (see the proof of [6, Lemma 1 (a)]). Furthermore, we can find $f \in ReR \subseteq RxR$ such that $e=fe=ef$. Noting that R is a chain ring and $I \neq R$, we can easily see that $RfR \subseteq R(1-f)^2R$. Thus we can write $e=\sum_i r_i(1-f)^2s_i$ with some r_i, $s_i \in R$, and therefore we get

$$x=e^2x=\sum_i \{er_i(1-f)\}\{(1-f)s_ix\} \in \langle I \cap N^* \rangle.$$

Lemma 1. (1) *If $[N^*,N^*]=0$, then $\langle N^* \rangle$ is a commutative nil subring. If, furthermore, R is semiprime, then $N^{*2}=0$.*

(2) *If $e \in E$, then $eR(1-e)R \subseteq \langle (eR \cup Re) \cap N^* \rangle$.*

Proof. (1) It suffices to prove the latter assertion. If $x,y \in N^*$, then $xyRxy=yRxyx=yRyx^2=0$, and hence $xy=0$.

(2) In fact, for any r, $s \in R$ we have

$$er(1-e)s=er(1-e)\cdot(1-e)se+er(1-e)s(1-e).$$

We are now in a position to state the second main theorem of this section.

Theorem 4. *Let R be a fully left and right idempotent chain ring. If $[N^*,N^*]=0$, then R is a simple ring without nontrivial idempotents.*

Proof. By Lemma 1 (1) and Proposition 3 (2), we see that R is a simple ring. Now, let e be a nonzero idempotent in R. Then, by Lemma 1 (2), $eR(1-e)R$ is a nilpotent right ideal, and hence $R(1-e)R=$

$ReR(1-e)R=0$, whence $eR=R=Re$ follows.

Corollary 3. *If R is a regular chain ring with $[N^*,N^*]=0$, then R is a division ring (and conversely).*

We conclude this paper with the following remarks.

Remark 2. If a regular chain ring R is a P.I. ring, then R is an Artinian simple ring. In fact, it is clear that the center of R is a field, and hence R is an Artinian simple ring by [9, Corollary 1.6.28]. Moreover, it is known that every regular ring R with $[N^*,N^*]=0$ is strongly regular (see [3, Lemma 2 (2)]).

Remark 3. A chain ring is called an *n-chain ring* if it has exactly n nontrivial ideals. Every simple ring with 1 is a fully left and right idempotent 0-chain ring. Given an algebra S with 1 over a field K, we denote by $S^{(1)}$ the set consisting of the matrices having the form

$$\begin{bmatrix} A & & & 0 \\ & k & & \\ & & k & \\ & & & \ddots \\ 0 & & & & \ddots \end{bmatrix}$$

where A is arbitrary finite matrix with entries in S and k is any element in K. If S is an n-chain ring then it is easy to see that the K-algebra $S^{(1)}$ is an $(n+1)$-chain ring. According to [2], there is a simple algebra R over $\boldsymbol{Q}$ which is a domain (with 1) but not a division ring (compare Theorem 4 with Corollary 3). Obviously, $R^{(n)}=(R^{(n-1)})^{(1)}$ is a fully left and right idempotent n-chain ring which is not a regular ring.

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