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ON A THEOREM OF MAYNE

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Throughout, R will represent an (associative) ring with center C. Let S be a subset of R. An (additive group) endomorphism T of R is said to be *centralizing* (resp. *skew-centralizing*) *on* S if $[s^T, s]=s^Ts-ss^T\in C$ (resp. $(s^T,s)=s^Ts+ss^T\in C$) for every $s\in S$. More generally, T is defined to be *semicentralizing on* S if $[s^T,s]\in C$ or $(s^T,s)\in C$ for every $s\in S$. In case $S=R$, we say simply T is *centralizing* (resp. *skew-centralizing*) or *semi-centralizing* according as so is T on R.

Recently, in [5], by making use of his previous result in [4], J.H. Mayne proved that if a prime ring R has a nontrivial ring automorphism T and a nonzero ideal U such that T is centralizing on U and $U^T\subseteq U$ then R is commutative. However, his proof is based on an unjustifiable assertion that T induces a ring automorphism on U. Incidentally, it should be mentioned that the result of [4] itself was claimed (even for a nontrivial surjective ring endomorphism) back in 1959 by M.F. Smiley [7, Remark 2].

In the present paper, we shall prove the following theorem which greatly generalizes [5, Theorem] and includes [1, Lemma] and [3, Corollary], as well.

Theorem 1. *Let U be a nonzero ideal of a prime ring R.*

(1) *Let T be a nontrivial ring endomorphism of R ($T\neq 1_R$). If T is semicentralizing on U, U^T is an ideal of R and $(U\cap U^T)^T$ is nonzero, then R is commutative.*

(2) *Let T be a nontrivial derivation of R ($T\neq 0_R$). If T is centralizing (resp. skew-centralizing) on U, then R is commutative.*

In preparation for proving our theorem, we state first several lemmas.

Lemma 1. *Let R be a prime ring, and I a right ideal of R.*

(1) *If I is nonzero and commutative, then R is commutative.*

(2) *Let T be a ring endomorphism of R. If I is nonzero and T is trivial on I, then T is itself trivial.*

(3) *Let T be a derivation of R. If I is nonzero and T is trivial on I, then T is itself trivial.*

(4) *Let T be a nontrivial derivation of R, and x an element of R. If $xR^T=0$ then $x=0$.*

(5) *If there exists a positive integer n such that $x^n=0$ for all $x \in I$, then $I=0$.*

Proof. (2), (3) and (4) are respectively [5, Lemmas 3, 2] and [6, Lemma 1] with routine proofs. (1) is [5, Lemma 4] and (5) is immediate by [2, Lemma 1.1]. However, for the sake of self-containedness, we prove (1) and (5).

(1) Given $a, b \in I$ and $x, y \in R$, we have $ab[x,y]=abxy-bayx=bxay-aybx=0$, namely $I^2[x,y]=0$. Hence, $[x,y]=0$ for all $x, y \in R$.

(5) We proceed by induction on n. First, we claim that $aI=0$ for any $a \in I$ with $a^2=0$. Let $A=aI$ and $S=\{x \in A \mid xA=0\}$. As is easily seen, S is a prime ideal of A. Furthermore, since $(ay)^{n-1}aI=(a+ay)^nI=0$ for any $y \in R$, we see that $x^{n-1} \in S$ for all $x \in A$. Hence, by induction hypothesis, $A/S=0$, i.e., $A^2=0$, whence it follows that $aI=0$. Now, let $W=\{x \in I \mid xI=0\}$. Then W is a prime ideal of I. Since the above claim tells us that $x^{n-1} \in W$ for all $x \in I$, our induction hypothesis shows $I=W$, i.e., $I^2=0$. Hence we have $I=0$.

Lemma 2. *Let $T: x \to x'$ be an endomorphism of R, and U an additive subgroup of R. Let $[U]=\{u \in U \mid [u',u] \in C\}$ and $(U)=\{u \in U \mid (u',u) \in C\}$.*

(1) *Let $u, v \in [U]$ (resp. (U)). Then $u+v \in [U]$ (resp. (U)) if and only if $u-v \in [U]$ (resp. (U)).*

(2) *If $v \in (U)$, then $[v',v^2]=[v,v'^2]=0$.*

Proof. (1) follows from $[u'-v',u-v]=-[u'+v',u+v]+2([u',u]+[v',v])$ (resp. $(u'-v',u-v)=-(u'+v',u+v)+2((u',u)+(v',v))$), and (2) is obvious by $[x,y^2]=[(x,y),y]$.

Lemma 3. *Let $T: x \to x'$ be a ring endomorphism of a prime ring R of characteristic not 2 which is semicentralizing on a nonzero ideal U, and let $[U]$, (U) be as in Lemma 2.*

(1) *If $v \in U \backslash [U]$, then $v^2v'^2=v'^2v^2=0$.*

(2) *If U^T is a nonzero ideal of R and $[U] \neq U$, then there is no positive integer n such that $v'^n=0$ for all $v \in U \backslash [U]$.*

Proof. (1) By Lemma 2 (2), we have

$$[v'^2+v',v^2+v]=[v'^2-v',v^2-v]=[v',v] \notin C,$$

which means that $v^2+v \notin [U]$ and $v^2-v \notin [U]$. Then, by Lemma 2 (1), $(v^2+v)-(v^2-v)=2v \notin [U]$ shows that $2v^2=(v^2+v)+(v^2-v) \in (U)$, and so $v^2 \in (U)$. Hence, by Lemma 2 (2), $2v'^2v^2=(v'^2,v^2)\in C$, i.e., $v'^2v^2\in C$. Furthermore, again by Lemma 2 (2),

$$0=v'^2[v^2+v',(v^2+v)^2]=2v'^2[v',v^3]=2v'^2v^2[v',v],$$

i.e., $v'^2v^2[v',v]=0$. Since $v'^2v^2\in C$ and R is prime, $[v',v]\neq 0$ implies that $v^2v'^2=v'^2v^2=0$.

(2) Suppose that $v'^n=0$ for all $v\in U\backslash[U]$. We shall show $v'=0$, which contradicts $v\notin[U]$. In order to see this, it suffices to show that $v'^{n-1}=0$ (if $n>1$). Let u be an arbitrary element of U. If $uv^{n-1}\notin[U]$ then $(u'v'^{n-1})^n=0$ by assumption. Next, suppose that $uv^{n-1}\in[U]$. Since

$$(uv^{n-1}+v)+(uv^{n-1}-v)=2uv^{n-1}\in[U]$$

and

$$(uv^{n-1}+v)-(uv^{n-1}-v)=2v\notin[U],$$

we see that either $uv^{n-1}+v\notin[U]$ or $uv^{n-1}-v\notin[U]$ (Lemma 2 (1)). Hence, either

$$(u'v'^{n-1})^{n+1}=u'v'^{n-1}(u'v'^{n-1}+v')^n=0$$

or

$$(u'v'^{n-1})^{n+1}=u'v'^{n-1}(u'v'^{n-1}-v')^n=0.$$

We have therefore seen that $(u'v'^{n-1})^{n+1}=0$ for all $u\in U$ and $v\in U\backslash[U]$. Since $U^Tv'^{n-1}$ is a nil left ideal of bounded index, we get $v'^{n-1}=0$ by Lemma 1 (5).

Corollary 1 (cf. [3, Theorem]). *Let T be a ring endomorphism of a prime ring R which is semicentralizing on a nonzero ideal U. If U^T is a nonzero ideal of R, then T is centralizing on U.*

Proof. We keep the notations in Lemma 2. If R is of characteristic 2, then $[x',x]=(x',x)$, and therefore T is centralizing on U. So, we assume henceforth that R is of characteristic not 2. Suppose $[U]\neq U$, and choose arbitrary $v\in U\backslash[U]$. Given $u\in U$, by making use of Lemma 2 (1) we can easily see that

$$uv^2u'v'^4+v^2u'v'^4=[uv^2+v^2,(uv^2+v^2)']v'^2=(uv^2+v^2,(uv^2+v^2)')v'^2.$$

Hence,

$$uv^2u'v'^4+v^2u'v'^4=c_1v'^2 \text{ with some } c_1\in C.$$

Similarly, considering uv^2-v^2 instead of uv^2+v^2, we get

$$uv^2u'v'^4 - v^2u'v'^4 = c_2v'^2 \text{ with some } c_2 \in C.$$

From those above, we obtain $2v^2u'v'^4=(c_1-c_2)v'^2$, and hence $2v^4u'v'^4=0$ again by Lemma 3 (1). This proves $v^4U^Tv'^4=0$, whence it follows that $v'^4=0$. But, this is impossible by Lemma 3 (2). We have thus proved that $[U]=U$.

Lemma 4. *Let $T: x \to x'$ be a derivation of a prime ring R of characteristic not 2 which is semicentralizing on a nonzero ideal U, and let $[U]$, (U) be as in Lemma 2.*

(1) *If $v \in U \backslash [U]$, then $(v^2)'=0$ and $v^2v'=v'v^2=0$.*

(2) *If $C \cap U=0$ and $v \in U \backslash [U]$, then $v'^3=0$ and $v^2 \neq 0$.*

(3) *If $C \cap U$ is nonzero, then T is centralizing on U.*

Proof. (1) Since $(v^2)'=(v',v) \in C$ and $[v',v^2]=0$ by Lemma 2 (2), we have

$$[(v^2+v)',v^2+v]=[(v^2-v)',v^2-v]=[v',v] \notin C,$$

which means that $v^2+v \notin [U]$ and $v^2-v \notin [U]$. Then, by Lemma 2 (1), $(v^2+v)-(v^2-v)=2v \notin [U]$ shows that $2v^2=(v^2+v)-(v^2-v) \in (U)$, and so $v^2 \in (U)$. Hence, $2(v^2)'v^2=((v^2)',v^2) \in C$, i.e., $(v^2)'v^2 \in C$. Furthermore, by Lemma 2 (2),

$$0=(v^2)'[(v^2+v)',(v^2+v)^2]=2(v^2)'[v',v^3]=2(v^2)'v^2[v',v],$$

i.e., $(v^2)'v^2[v',v]=0$. Since $(v^2)'v^2 \in C$ and R is prime, $[v',v] \neq 0$ implies $(v^2)'v^2=0$. Recalling here that $(v^2)' \in C$, we get $(v^2)'=0$. Since $v^2+v \notin [U]$, we have also $0=((v^2+v)^2)'=((v^2+v)',v^2+v)=(v',v^2+v)=2v'v^2$, and so $v'v^2=v^2v'=0$ by Lemma 2 (2).

(2) Observe that $vv'=-v'v$ and $uu'=\pm u'u$ for every $u \in U$. We prove first that $v^2 \neq 0$. In fact, if $v^2=0$ then for any $x \in R$ we have

$$vxv'v+xvxv'v=\{(v+xv)(v+xv)' \pm (v+xv)'(v+xv)\}v=0.$$

Replace x by $-x$ in the above to get $-vxv'v+xvxv'v=0$. Hence $vRv'v=0$, and therefore $v'v=0$. But this contradicts $v \notin [U]$.

Next, we claim that $vv'^2=0$. Noting that $v^2v'=0$ by (1), for any $x \in R$ we have

$$-v^2xvv'^2-vxv^2xvv'^2=\{(v+vxv)(v+vxv)' \pm (v+vxv)'(v+vxv)\}vv'=0,$$

and similarly $v^2xvv'^2-vxv^2xvv'^2=0$. Hence $v^2Rvv'^2=0$, and therefore $vv'^2=0$ by $v^2 \neq 0$.

Now, for any $x \in R$ we have

$$vxv'^3+xvxv'^3=\{(v+xv)(v+xv)'\pm(v+xv)'(v+xv)\}v'^2=0,$$

and similarly $-vxv'^3+xvxv'^3=0$. Hence $vRv'^3=0$, and therefore $v'^3=0$.

(3) Suppose U contains an element v not contained in $[U]$. Choose an arbitrary nonzero $c\in C\cap U$. Because $c'\in C$, we have $[v'+c',v+c]=[v',v]\notin C$, and so $v+c\notin[U]$. Then, by (1),

$$0=[\{(v+c)^2\}',v]=[2cv'+2c'v+(c^2)',v]=2c[v',v],$$

i.e., $[v',v]=0$. This contradiction proves that $[U]=U$.

Corollary 2. *Let $T:x\to x'$ be a derivation of a prime ring R, and U a nonzero ideal of R.*

(1) *If T is skew-centralizing on U, then it is centralizing on U.*

(2) *If T is semicentralizing on U and U^T is a left (resp. right) ideal of R, then T is centralizing on U.*

Proof. We may assume that $T\neq 0_R$ and R is of characteristic not 2.

(1) According to Lemma 4 (3), it suffices to show that $C\cap U$ is nonzero. Suppose, to the contrary, that $C\cap U=0$. Then, for any $u\in U$ and $x\in R$,

$$(u^2x+uxu)'=\{(u+ux)^2-u^2-(ux)^2\}'=0$$

and

$$(xu^2+uxu)'=\{(u+xu)^2-u^2-(xu)^2\}'=0.$$

From those above, we readily obtain $[x,u^2]'=0$. This means that $DT=0$, where D is the inner derivation of R effected by u^2. Now, suppose that D is nontrivial. Let a, b and c be arbitrary elements of R. Obviously,

$$(*)\qquad a'b^D+a^Db'=(ab)^{DT}=0.$$

Noting that $b^{D^2}c'=(b^Dc)^{DT}=0$ and $b^Dc'=-b'c^D$, we have

$$0=(ab^D)'c^D+(ab^D)^Dc'=a'b^Dc^D+a^Db^Dc'=(a'b^D-a^Db')c^D,$$

namely $(a'b^D-a^Db')R^D=0$. Hence, $a'b^D-a^Db'=0$ by Lemma 1 (4). Combining this with $(*)$, we get $a'R^D=0$. Again by Lemma 1 (4), $a'=0$ for all $a\in R$, i.e., $T=0$. This contradiction proves $D=0$, which tells us that $u^2=0$ for all $u\in U$. But, this is impossible by Lemma 1 (5).

(2) Suppose, to the contrary, that U contains an element v not contained in $[U]$. In view of Lemma 4 (3), it suffices to consider the case that $C\cap U=0$. Let u be an arbitrary element of U. If $uv^2\notin[U]$ then $(u'v^2)^3=(uv^2)'^3=0$ by Lemma 4. On the other hand, if $uv^2\in[U]$ then it

is easy to see that either $v+uv^2 \notin [U]$ or $v-uv^2 \notin [U]$ (Lemma 2 (1)). Hence, by Lemma 4,

$$(u'v^2)^4=(u'v^2)\{(u'v^2)^3+v'(u'v^2)^2+v'^2(u'v^2)\}=u'v^2(v+uv^2)'^3=0$$

or

$$(u'v^2)^4=(u'v^2)\{(u'v^2)^3-v'(u'v^2)^2+v'^2(u'v^2)\}=u'v^2(v-uv^2)'^3=0.$$

Therefore $(u'v^2)^4=0$ for all $u \in U$. Now, choose $r \in R$ such that $r' \neq 0$. Then, $r'u=(ru)'-ru' \in U'$ for all $u \in U$, i.e., $r'U \subseteq U'$, and hence U' contains a nonzero ideal $Rr'U$. Since $Rr'Uv^2$ is a nil left ideal of bounded index, we get $v^2=0$ by Lemma 1 (5). But, this is impossible by Lemma 4 (2).

We are now ready to complete the proof of our theorem.

Proof of Theorem 1. For the convenience of notation, let us write $x^T=x'$.

(1) We put $W=U \cap U'$. Obviously, U' is a prime ring and W' is a nonzero ideal of U'. It is well known that $C' \subseteq C$. According to Corollary 1, T is centralizing on U. Now, by Jacobi's identity, we have $[[u,u''],u']=0$ for all $u \in U$, and so

$$\begin{aligned}[u,u'][u'',u'] &= [u'[u,u'']+[u,u']u''+[u,u']u',u'] \\ &= [[u,u'u'']+[uu',u'],u'] \\ &= [[u+uu',(u+uu')'],u']=0.\end{aligned}$$

Hence, $[x,x']=0$ for all $x \in U'$. Linearizing $[x,x']=0$ gives $[x,y']=[x',y]$ for all x, $y \in U'$, and then

$$(x-x')[x,y']=x[x,y']-[x,x'y']=x[x',y]-[x',xy]=0.$$

Hence, noting that $z'[x,y']=[x,(zy)']-[x,z']y'$, we see that $(x-x')W'[x,y']=0$ (x, $y \in W$). Then, since U' is a prime ring, we have $W=V_W(W') \cup K$, where $V_W(W')$ is the centralizer of W' in W and $K=\{x \in W \mid x'=x\}$. In view of Lemma 1 (2), $W \neq K$, and so $W=V_W(W')$ (by Brauer's trick). In particular, the nonzero ideal $W \cap W'$ of U' is commutative. Now, U' is commutative by Lemma 1 (1), and hence so is R by the same lemma.

(2) In view of Corollary 2 (1), T is centralizing on U. We consider the ring $R_1=\{\begin{pmatrix} x & y \\ 0 & x\end{pmatrix} \mid x,\ y \in R\}$ with center $C_1=\{\begin{pmatrix} x & y \\ 0 & x\end{pmatrix} \mid x,\ y \in C\}$, where R is regarded as a subring of R_1 in an obvious way. As is easily seen, T gives rise to a ring homomorphism $x \to x^*=\begin{pmatrix} x & x' \\ 0 & x\end{pmatrix}$ of R into R_1 and $[u^*,u]$

$\in C_1$ for all $u \in U$. First, we claim that $[u',u]=0$, or equivalently $[u^*,u]=0$, for all $u \in U$. If R is of characteristic 2, then

$$\begin{aligned}0&=[[u+uu',(u+uu')'],u]=[[uu',u']+[u,(uu')'],u]\\&=[u',u]^2+[u[u,u''],u].\end{aligned}$$

Since $[u,u'']=[u,u']' \in C$, the last shows that $[u',u]^2=0$, and hence $[u',u]=0$. On the other hand, if R is of characteristic not 2, then

$$4\begin{pmatrix}0 & u^2[u',u]\\ 0 & 0\end{pmatrix}=2(u^*,u)[u^*,u]=[(u^2)^*,u^2]\in C_1,$$

i.e., $u^2[u',u]\in C$. Hence, $0=[u',u^2[u',u]]=2[u',u]^2u$, and therefore $[u',u]=0$.

Now, linearizing $[u^*,u]=0$ gives $[u,v^*]=[u^*,v]$ for all $u,\ v\in U$, and then

$$(u-u^*)[u,v^*]=u[u,v^*]-[u,u^*v^*]=u[u^*,v]-[u^*,uv]=0.$$

Hence, noting that $x^*[u,v^*]=[u,(xv)^*]-[u,x^*]v^*$ $(u,\ v\in U,\ x\in R)$, we get $(u^*-u)\,x^*[u,v^*]=0$, which becomes $u'x[u,v]=0$, i.e., $u'R[u,v]=0$. Thus, we get $U=V_U(U)\cup K$, where $K=\{u\in U \mid u'=0\}$. Since $U\neq K$ by Lemma 1 (3), U coincides with its center, and therefore R is commutative by Lemma 1 (1).

Corollary 3. *Let U be a nonzero ideal of a prime ring R.*

(1) *Let T be a nontrivial ring endomorphism of R. If T induces a semicentralizing endomorphism of U, U^T is an ideal of U and $U^{T^2}\neq 0$, then R is commutative.*

(2) *Let T be a nontrivial derivation of R. If T induces a centralizing (resp. skew-centralizing) derivation of U, then R is commutative.*

Proof. U is a prime ring and T is nontrivial on U (Lemma 1 (2) and (3)). Hence, U is commutative by Theorem 1, and therefore so is R by Lemma 1 (1).

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Added in proof. After the submission of this paper, the authors received from Prof. J.H. Mayne an erratum sheet that corrects the proofs of [5, Theorem and Corollary] and a copy of his paper entitled "Centralizing mappings of prime rings" (submitted to Canad. Math. Bull.), where he has improved [5, Theorem].