

RINGS WHOSE MULTIPLES ARE DIRECT SUMMANDS

DINH VAN HUYNH

Throughout, A will represent an (associative) ring, A^+ the additive group of A , and D the maximal divisible ideal of A . We shall use the symbol \boxplus (resp. \oplus) for a (ringtheoretic) direct sum (resp. grouptheoretic direct sum). Let Π be the set of prime numbers.

The purpose of this note is to prove the following theorem, which gives a weak answer of a problem raised by Szász [1, Problem 79].

Theorem 1. *The following are equivalent :*

- 1) *For each integer n , nA is a direct summand of A .*
- 2) *A contains an ideal I consisting of torsion elements with square-free order such that $D \cap I = 0$ and either $A = D \boxplus I$ or $A^+ / (D \boxplus I)^+$ is divisible.*

In advance of proving our theorem, we state the next

Lemma 1. (1) *Let $p \in \Pi$, and U a subgroup of A^+ with $pU = U$. If u is a non-zero element of U with $pu = 0$, then there exists a quasicyclic subgroup of U containing u ; $u \in D$.*

(2) *Let P be a subset of Π . Suppose that $A = pA \boxplus A_p$ or $A = pA$ according as $p \in P$ or $p \in \Pi \setminus P$. If $p_1, \dots, p_k \in P$ are different, then $A = A_{p_1} \boxplus \dots \boxplus A_{p_k} \boxplus p_1 p_2 \dots p_k A$, where $q(p_1 \dots p_k A) = p_1 \dots p_k A$ for every $q \in \{p_1, \dots, p_k\} \cup \Pi \setminus P$. In particular, if P is finite then $A = D \boxplus (\boxplus_{p \in P} A_p)$.*

Proof. (1) Straightforward.

(2) Obviously, $pA = p^2A$ and $pA_p = 0$ for every $p \in P$. The rest of the proof is a routine.

Proof of Theorem 1. 1) \Rightarrow 2). Let P be the set of all $p \in \Pi$ with $pA \neq A$; $A = pA \boxplus A_p$. If P is empty then $A = D$. Now, let P be non-empty, and $p \in P$. Then $pA_p = 0$ and $pA = p^2A$. By Lemma 1 (2), we readily see that $B = D + \sum_{p \in P} A_p = D \boxplus (\boxplus_{p \in P} A_p)$. Furthermore, since $A = pA + B$ for every $p \in \Pi$, A^+ / B^+ is divisible provided $A \neq B$.

2) \Rightarrow 1). We set $B = D \boxplus I = D \boxplus (\boxplus_{p \in P} A_p)$, where P is a subset of

Π and A_p is an ideal of A with $pA_p=0$. Further, we may assume that $n = p_1^{m_1} \cdots p_r^{m_r} q_1^{n_1} \cdots q_s^{n_s}$ with different $p_i \in P$ and $q_j \in \Pi \setminus P$.

Let $p \in \Pi$. Since $A=B$ or A^+/B^+ is divisible, we have

$$A = pA + B = \begin{cases} pA + pB = pA & \text{if } p \notin P \\ pA + pB + A_p = pA + A_p & \text{if } p \in P. \end{cases}$$

Suppose that $p \in P$ and $pA \cap A_p$ contains a non-zero element a . Then, because $pA = p^2A$, Lemma 1 (1) forces a contradiction $a \in D \cap A_p = 0$. Thus, we have shown that $A = pA$ or $A = pA \boxplus A_p$ according as $p \notin P$ or $p \in P$. Then, by Lemma 1 (2), $A = A_{p_1} \boxplus \cdots \boxplus A_{p_r} \boxplus p_1 \cdots p_r A = A_{p_1} \boxplus \cdots \boxplus A_{p_r} \boxplus nA$.

Corollary 1. *Suppose that A^+ contains no quasicyclic subgroups. If nA is a grouptheoretic direct summand of A^+ for each integer n , then nA is a direct summand of A .*

Proof. Let P be the set of all $p \in \Pi$ with $pA \neq A$. Suppose now that P is non-empty, and let $p \in P$. Then $A^+ = pA \oplus A_p$, and so $pA_p = 0$ and $pA = p^2A$. Now, let $a, b \in A_p$, and $ab = c + c'$ with $c \in pA$, $c' \in A_p$. Then $pc = 0$, and Lemma 1 (1) shows that $c = 0$. This proves that A_p is a subring of A , and therefore $A = pA \boxplus A_p$. Hence, R has the property 1) (see the last part of the proof of Theorem 1).

Remark 1. Let P be an infinite subset of Π , and A_p rings with $pA_p = 0$ ($p \in P$). Then $A_0 = \boxplus_{p \in P} A_p$ has the property 1). Also, the complete direct sum A of A_p ($p \in P$) has the same and A^+/A_0^+ is divisible. Now, decompose P into the disjoint infinite subsets $P^{(i)}$ ($i = 1, 2, \dots$), and denote by $A^{(i)}$ the complete direct sum of A_p ($p \in P^{(i)}$). Then $B = \boxplus_{i=1}^{\infty} A^{(i)}$ has the property 1) and $A \supseteq B \supseteq A_0$.

Remark 2. Let A be a ring with property 1). If the maximal torsion ideal T of A is a grouptheoretic direct summand of A^+ and contains no quasicyclic subgroups, then T is a direct summand of A .

REFERENCE

[1] F. SZÁSZ: Radicals of Rings, John Wiley & Sons, Chichester-New York-Brisbane-Toronto, 1981

MARTIN-LUTHER-UNIVERSITÄT HALLE WITTENBERG
SEKTION MATHEMATIK
DDR-4020 HALLE (S.)
UNIVERSITÄTSPLATZ 6

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