ON CONTINUITY OF INJECTIVITY RADIUS FUNCTION

Dedicated to Prof. S. Sasaki on his 70th birthday

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0. Introduction. Let M be a compact connected C^∞-manifold. For a fixed riemannian structure g on M we define the injectivity radius $i_g(M)$ of g by

Sup$\{r > 0 : \mathrm{Exp}_x : B_r(o_x) \to M$ is a diffeomorphism for every x of $M\}$, where Exp_x denotes the exponential map at x, $B_r(o_x)$ an r-ball in T_xM centered at the origin. This gives a uniform estimate of the size of domains over which normal coordinates are valid. Now we consider the space $\mathfrak{M}$ of C^3-riemannian structures g on M with the C^2-topology. We may consider $g \to i_g(M)$ as a function on $\mathfrak{M}$. P. Ehrlich ([E]) proved

Theorem. $g \to i_g(M)$ *is a continuous function.*

Namely when $g_n \to g_o$ (w.r.t. C^2-topology) we have to show $\lim \sup i_{g_n}(M) \leq i_{g_o}(M)$ and $\lim \inf i_{g_n}(M) \geq i_{g_o}(M)$. I applied the first inequality in a previous paper ([S]). Since Ehrlich's proof is rather complicated, I presented a simple proof of the first inequality in my first draft of the above paper. Then the referee of my paper pointed out that the first inequality may be proved much more simply by using the Busemann type argument. Here I will give a very simple proof of the second inequality, which is a more difficult one. We include a proof of the first inequality for completeness and I hearty appreciate the referee of my paper ([S]).

1. Preliminaries. Let g be a riemannian metric of class C^3 on a compact connected C^∞-manifold M. We denote by d_g the distance function induced from g. Firstly we recall two fundamental facts about the injectivity radius $i_g(M)$:

(1.1) $i_g(M)$ is given by the minimum of the shortest distance to the first conjugate points along geodesics and half the length of the shortest closed geodesic.

(1.2) Take $x, y \in M$ with $d_g(x,y) = i_g(M)$ and let $c : [0, i_g(M)] \to M$ be a minimal geodesic from x to y parametrized by arc-length. If y is not conjugate to x along any minimal geodesic from x to y, $c|[0, 2i_g(M)]$ defines a closed geodesic.

Next we recall some fundamental facts from the ordinary differential equations:

Lemma 1.3. *Let N be a C^2-riemannian manifold, X a C^1-vector field on N and $x(t), 0 \leq t \leq R$ an integral curve of X in a bounded domain $D \subset N$. Then there exist positive constants C, ε_o, δ_o depending on D, X, $x(0)$ and R with the following property: Take any $0 < \varepsilon \ (\leq \varepsilon_o)$, $0 < \delta \ (\leq \delta_o)$ and a C^1-vector field Y on N such that $\sup_{x \in D} |X_x - Y_x| < \delta$. Then for any integral curve $y(t)$ of Y with $d(x(0), y(0)) < \varepsilon$ we have $y(t) \in D$, $0 \leq t \leq R$ and $d(x(t), y(t)) < C(\varepsilon + \delta t)$.*

Proof. Take a local chart (φ, U), $D \supset \bar{U}$, $U \supset \bar{V}$, $V \supset x([0, R])$ such that $x(t)$, $0 \leq t \leq R$ lie on a coordinate axis and $\varphi(\bar{V})$ is a convex subset in $R^{\dim N}$ (e.g., take Fermi coordinate of $x(t)$, $0 \leq r \leq R$ if $X_{x(0)} \neq 0$ and normal coordinate of $x(0)$ if $X_{x(0)} = 0$). There exists a constant $A > 1$ such that

$$1/A^2 \cdot (\delta_{ij}) \leq (g_{ij}) \leq A^2 (\delta_{ij})$$

on $\bar{V}$, where (g_{ij}) denotes the riemannian metric. Denoting the euclidean norm and riemannian norm by $\| \cdot \|$ and $| \cdot |$ respectively, we have then $\|x(0) - y(0)\| \leq Ad(x(0), y(0))$ if $y(0)$ is in a convex neighborhood of $x(0)$ and $\|X_x - Y_x\| \leq A|X_x - Y_x| < A\delta$ for $x \in \bar{V}$. Since X is of class C^1 and $\varphi(\bar{V})$ is convex, there exists a positive constant B such that $\|X_x - X_y\| \leq B\|x - y\|$ on $\bar{V}$. First assuming that $y(t) \in V$, $0 \leq t \leq R$ and $d(x(0), y(0)) < \varepsilon$, we have

$$\|\dot{x}(t) - \dot{y}(t)\| \leq \|X_{x(t)} - X_{y(t)}\| + \|X_{y(t)} - Y_{y(t)}\| \leq A\delta + B\|x(t) - y(t)\|.$$

From this we easily see that

$$\{\|x(t) - y(t)\| e^{-Bt}\}' < A\delta e^{-Bt} \leq A\delta \text{ as far as } x(t) \neq y(t).$$

Putting $s(t) := \text{Inf}\{s : x(s) \neq y(s) \text{ for } (0 \leq) s \leq t\}$, we get

$$\|x(t) - y(t)\| e^{-Bt} \leq \int_{s(t)}^{t} \{\|x(t) - y(t)\| e^{-Bt}\}' dt + \|x(s(t)) - y(s(t))\| e^{-Bs(t)}$$
$$\leq A\delta t + \|x(0) - y(0)\|,$$

namely

$$\|x(t) - y(t)\| \leq e^{Bt} A(\delta t + d(x(0), y(0))).$$

Setting $C := \text{Max}\{1, A^2 e^{BR}\}$, we have

$$d(x(t), y(t)) \leq C(\delta t + d(x(0), y(0)))$$

as far as $|X_x - Y_x| < \delta$ and $y(t) \in V$.

Next choose ε_o, δ_o so that $C(\delta_o R + \varepsilon_o) < a$, where a is a positive number such that $B_{x(t)}(a) := \{y : d(x(t), y) < a\} \subset V$ for all $0 \leq t \leq R$. Then we will be done if we show that $t_1 := \sup\{t : y(s) \in V$ for $0 \leq s \leq t\}$ equals R for Y satisfying the assumption of the lemma. Clearly $t_1 > 0$ and assume that $t_1 < R$. Then by continuity we have $d(x(t_1), y(t_1)) \leq C(\delta t_1 + \varepsilon) < a$. Then $y(t_1) \in V$ and consequently $y(t) \in V$ for $t > t_1$, $|t - t_1|$ sufficiently small, a contradiction.

Now on a C^∞-manifold we mean by C^r-topology the topology of uniform convergence on compact subsets of derivatives up to order r.

Corollary 1.4. *Let X_n ($n = 1, 2, \cdots$), X be C^1-vector fields on a C^∞-manifold N. Assume that X is complete and $X_n \to X$ (w.r.t. C^0-topology). Let ϕ_t^n, ϕ_t be flows generated by X_n, X respectively. If $x_n \to x$, $t_n \to t$ then we have $\lim\limits_{n \to \infty} \phi_{t_n}^n(x_n) = \phi_t(x)$.*

Proof. Choose a riemannian metric on N, $R > t_n$, t. Let C, ε_o, δ_o be as in Lemma 1.3 for X, x, R. For any $\varepsilon_1 > 0$ take $(\varepsilon_o \geq)\varepsilon > 0$, $(\delta_o \geq)\delta > 0$ so that $C(\varepsilon + \delta R) < \varepsilon_1/2$. Then for sufficiently large n we have $d(x_n, x) < \varepsilon$, $|X_n - X| < \delta$ on a compact domain D whose interior contains $\phi_t(x)$, $0 \leq t \leq R$ and $d(\phi_{t_n}(x), \phi_t(x)) < \varepsilon_1/2$. Then from Lemma 1.3 we get for such n

$$d(\phi_{t_n}^n(x_n), \phi_t(x)) \leq d(\phi_{t_n}^n(x_n), \phi_{t_n}(x)) + d(\phi_{t_n}(x), \phi_t(x))$$
$$\leq C(\varepsilon + \delta t_n) + \varepsilon_1/2 < \varepsilon_1.$$

Now to prove the continuity of $g \to i_g(M)$ we take the geodesic flow view point. For a riemannian structure g on M we denote by ϕ_t^g the geodesic flow on the tangent bundle $\pi : TM \to M$. Namely for $v \in TM - \{0\}$, $t \to \pi \circ \phi_t^g(v)$ is a geodesic emanating from πv with the initial direction v which will be denoted by $c_v^g(t)$.

We denote by $U(M, g)$ the unit tangent bundle with respect to g.

Lemma 1.5. *Let (M, g_o) be a complete C^2-riemannian manifold and g_n ($n = 1, 2, \cdots$) a sequence of C^2-riemannian metrics on M such that $g_n \to g_o$ (w.r.t. C^1-topology). Then for $v_n \in U(M, g_n)$, $v_n \to v \in U(M, g_o)$ w.r.t. the topology of TM and $t_n \to t$ we have*

(i) $\lim\limits_{n \to \infty} \phi_{t_n}^{g_n}(v_n) = \phi_t^{g_o}(v)$

(ii) $\lim\limits_{n \to \infty} c_{v_n}^{g_n}(t_n) = c_v^{g_o}(t)$.

Proof. Let S^g be the geodesic spray with respect to g which is a C^1-vector field on TM given by

$$S^g_{(x,\xi)} = \sum_i \xi^i \ \partial/\partial x^i - \sum_{i,j,k} \Gamma^i_{jk} \xi^j \xi^k \ \partial/\partial \xi^i,$$

in terms of local coordinates of M. Then clearly $S^{g_n} \to S^{g_0}$ (w.r.t. C^0-topology) when $g_n \to g_0$ (w.r.t. C^1-topology). Since ϕ^g_t is a flow generated by S^g we get the first assertion from Corollary 1.4. The second assertion is trivial because $c^g_v(t) = \pi \circ \phi^g_t(v)$.

Lemma 1.6. *Let (M,g_0) be a complete C^3-riemannian manifold and $g_n \to g_0$ (w.r.t. C^2-topology) be a sequence of C^3-riemannian metrics on M. Then if $v_n \in U(M,g_n) \to v \in U(M,g_0)$ in TM, $w_n \in T_{v_n}TM \to w \in T_v TM$ in TTM and $t_n \to t$, we have*

(i) $\displaystyle \lim_{n \to \infty} (\phi^{g_n}_{t_n})_* w_n = (\phi^{g_0}_t)_* w$

(ii) $\displaystyle \lim_{n \to \infty} \pi_*(\phi^{g_n}_{t_n})_* w_n = \pi_*(\phi^{g_0}_t)_* w.$

Proof. First note that the following easily proved fact: Let $X = \sum_i X^i \ \partial/\partial x^i$ be a C^2-vector field on a C^∞-manifold N and ϕ_t a flow generated by X. Then the C^1-vector field X^* on TN defined from a flow $(\phi_t)_*: TN \to TN$ is given by

$$X^*_{(x,v)} = \sum_i X^i(x) \ \partial/\partial x^i + \sum_i v \cdot X^i \ \partial/\partial \xi^i$$

with respect to the adapted chart on TN.

Applying this to the geodesic spray on TM, we have with respect to the adapted local coordinate $(x^i, \xi^i, y^i, \eta^i)$ on TTM,

$$S^*_{(x,\xi,y,\eta)} = \sum_i \xi^i \ \partial/\partial x^i - \sum_{i,j,k} \Gamma^i_{jk}(x) \xi^j \xi^k \ \partial/\partial \xi^i$$

$$+ \sum_i \eta^i \ \partial/\partial y^i - \sum_{i,j,k,l} \left(2\Gamma^i_{jk}(x) \xi^j \eta^k + \frac{\partial \Gamma^i_{jk}(x)}{\partial x^l} y^l \xi^j \xi^k \right) \partial/\partial \eta^i.$$

Namely if (M,g) is a C^3-riemannian manifold, S^* is a C^1-vector field on TTM. Moreover $(S^{g_n})^* \to (S^{g_0})^*$ (w.r.t. C^0-topology) if $g_n \to g_0$ (w.r.t. C^2-topology) and Lemma 1.6 also follows from Corollary 1.4.

Lemma 1.7. *Let M be a compact manifold and g_n, g_0 be C^0-riemannian metrics such that $g_n \to g_0$ (w.r.t. C^0-topology), namely $(1-\varepsilon_n)g_0 \leq g_n \leq (1+\varepsilon_n)g_0$ with $\varepsilon_n \to 0$. Then we have*

$$(1-\varepsilon_n)d_{g_0}(x,y) \leq d_{g_n}(x,y) \leq (1+\varepsilon_n)d_{g_0}(x,y)$$

for $x, y \in M$.

Proof. This follows easily from

$$(1 - \varepsilon_n) L_{g_0}(c) \leq L_{g_n}(c) \leq (1 + \varepsilon_n) L_{g_0}(c)$$

for any piecewise smooth curve connecting x and y, and the definition of the ditance.

Corollary 1.8. *Under the assumption of Lemma* 1.7 *we have*

$$\lim_{n \to \infty} d_{g_n}(M) = d_{g_0}(M),$$

where $d_g(M)$ *denotes the diameter of* M.

Proof. This is clear from the definition of the diameter, i.e., $d_g(M) := \max_{x, y \in M} d_g(x, y)$ and the lemma.

2. Proof of Theorem.

$1°$. $\limsup i_{g_n}(M) \leq i_{g_0}(M)$.

Put $R_n := i_{g_n}(M)$, $R := \limsup R_n$ and fix any $x \in M$. Then it suffiees to show that for any g_0-geodesic c_v which emanates from x with initial tangent vector $v \in U_x(M, g_0)$ we have $d_{g_0}(x, c_v(R)) \geq R$. Taking a subsequence we may also assume that $R_n \to R$. Recall that $c_v(t) = \pi \circ \phi_t^{g_0}(v)$. Let $c_{v_n}^n$ be the g_n-geodesic emanating from x with initial tangent vector $v_n := v / |v|_{g_n} \in U_x(M, g_n)$, namely $c_{v_n}^n(t) = \pi \circ \phi_t^{g_n}(v_n)$. Since $g_n \to g_0$ (w.r.t. C^1-topology), $v_n \to v$ in TM, $R_n \to R$ hold and we have from Lemmas 1.5 and 1.7

$$\begin{aligned}
R = \lim_{n \to \infty} R_n &= \lim_{n \to \infty} d_{g_n}(x, c_{v_n}^n(R_n)) \\
&\leq \lim_{n \to \infty} (d_{g_n}(x, c_v(R)) + d_{g_n}(c_v(R), c_{v_n}^n(R_n))) \\
&\leq d_{g_0}(x, c_v(R)) + \lim_{n \to \infty} (1 + \varepsilon_n) d_{g_0}(c_v(R), c_{v_n}^n(R_n)) \\
&= d_{g_0}(x, c_v(R)).
\end{aligned}$$

Remark 1. For this inequality we only have to assume that $g_n \to g_0$ (w.r.t. C^1-topology).

$2°$. $\liminf i_{g_n}(M) \geq i_{g_0}(M)$.

Taking a subsequence we may assume that $R_n := i_{g_n}(M) \to R$ and we have to show that $R \geq i_{g_0}(M)$. Choose points $x_n, y_n \in M$ with $d_{g_n}(x_n, y_n)$

$= R_n$ and let v_n be the g_n-unit initial tangent vector to a minimal g_n-geodesic $c_{v_n}^n$ connecting x_n and y_n. Again we may assume that $x_n \to x$, $y_n \to y$, $v_n \to v$ by taking subsequences if necessary. Firstly we show that R is positive. In fact if $R = 0$, we have $x = y$ by Lemma 1.7. Since $g_n \to g_o$ (w.r.t. C^2-topology) we can find a universal positive constant K such that sectional curvesures K_{g_n} of the metric g_n satisfy $K_{g_n} \leq K$, $K_{g_o} \leq K$ for sufficiently large n. Then there appears no conjugate point to x_n along g_n-geodesic $c_{v_n}^n$ up to the (arc-length) parameter value $\pi/\sqrt{K}$. Thus from (1.2) we see that $c_{v_n\|[0,2R_n]}^n$ are g_n-closed geodesics for sufficiently large n because $R_n \to 0$. Then for any $\varepsilon > 0$, $t \to c_{v_n}^n(t)$ are contained in an ε-ball $B_\varepsilon(x)$ (w.r.t. g_o) centered at x for sufficiently large n by Lemma 1.7. Then from Lemma 1.5 we see that g_o-geodesic $t \to c_v(t) = \pi \circ \phi_t^{g_o}(v)$ is also contained in $\overline{B_\varepsilon(x)}$. Taking $\varepsilon < i_{g_o}(M)/2$ we get a contradiction. Thus R is positive. Next we consider the following two cases:

Case 1. For infinitely many n, y_n is not conjugate to x_n along any minimal geodesic connecting x_n and y_n. Let $c_{v_n\|[0,R_n]}^n$ be a minimal g_n-geodesic from x_n to y_n with initial direction $v_n \in U_{x_n}(M,g_n)$. Then from (1.2) $c_{v_n\|[0,2R_n]}^n$ defines a closed geodsic, namely $\phi_{2R_n}^{g_n}(v_n) = v_n$. Again from Lemma 1.5 we have $\phi_{2R}^g(v) = v$, where v is a cluster point of v_n and belongs to $U_x(M,g_o)$. This means that there exists a closed g_o-geodesic of length $2R$ and we get $i_{g_o}(M) \leq R$ ((1.1)).

Case 2. For almost all n, y_n is conjugate to x_n along some minimal g_n-geodesic $c_{v_n[0,R_n]}^n$. First we recall a characterization of Jacobi fields from the geodesic flow view point.

Lemma 2.1. *Let (M,g) be a riemannian manifold and v, $w \in T_xM$. Regarding w as an element of $T_vT_xM(\subset T_vTM)$ via the canonical identification $T_vT_xM \simeq T_xM$, $Y(t) := \pi_*(\phi_t^g)_*w$ is a Jacobi field along a geodesic $t \to \pi \circ \phi_t^g(v)$ with the initial condition $Y(0) = 0$, $\nabla Y(0) = w$. Conversely any Jacobi field Y with $Y(0) = 0$ may be expressed in the above way.*

Proof. Put $\alpha(s,t) := \pi \circ \phi_t^g(sw + v)$. Then $t \to \alpha_s(t) := \alpha(s,t)$ are geodesics from x for all s and $Y(t) = \pi_*(\phi_t^g)_*w = \dfrac{\partial \alpha(s,t)}{\partial s}_{|s=0}$ is a Jacobi field.

Moreover we have $Y(0) = \partial/\partial s_{|s=0}\alpha(s,0) = 0$ and

$$\nabla Y(0) = \nabla_{\partial/\partial t|t=0}\frac{\partial \alpha(s,t)}{\partial s}_{|s=0} = \nabla_{\partial/\partial s|s=0}\frac{\partial \alpha(s,t)}{\partial t}_{|t=0} = \nabla_{\partial/\partial s|s=0}(sw + v) = w.$$

Converse is clear because the Jacobi field is uniquely determined by $Y(0)$ and $\nabla Y(0)$.

Now we return to the second case. We have non-trivial g_n-Jacobi fields Y_n along $c_{v_n}^n$ which vanish at 0 and R_n. We may assume that Y_n is g_n-perpendicular to $c_{v_n}^n$ and $w_n := \nabla Y_n(0) \in U_{x_n}(M, g_n)$. Then from Lemma 2.1 we have $Y_n(t) = \pi_*(\phi_t^{g_n})_* w_n$. Since $\{(v_n, w_n)\}$ is contained in a compact subset of $TM \times TM$, we may assume that $v_n \to v$, $w_n \to w$ and $R_n \to R$ by taking subsequences if necessary. Clearly $v, w \in U_x(M, g_0)$, $g_0(v, w) = 0$. Now from Lemma 1.6 we get

$$0 = Y_n(R_n) = \pi_*(\phi_{R_n}^{g_n})_* w_n \to \pi_*(\phi_R^{g_0})_* w = 0.$$

Then $Y(t) := \pi_*(\phi_t^{g_0})_* w$ is a g_0-Jacobi field along c_v which satisfies $Y(0) = 0$, $\nabla Y(0) = w$ ($\neq 0$) and $Y(R) = 0$. Namely y is conjugate to x along c_v and we get again $i_{g_0}(M) \leq R$ ((1.1)).

Remark 2. Define the injectivity radius $i_x(M, g)$ of (M, g) at x as

$$i_x(M, g) := \mathrm{Sup}\ \{r > 0 : \mathrm{Exp}_x : B_r(o_x) \to M \text{ is a diffeomorphism}\},$$

and the diameter from x as

$$d_x(M, g) := \mathrm{Max}_{y \in M}\ d(x, y).$$

Then by a similar argument we can show that the functions on $\mathfrak{M}$ defined by $g \to i_x(M, g)$ and $g \to d_x(M, g)$ is continuous.

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