Math. J. Okayama Univ. 25 (1983), 87—90

ON THE TWO DECOMPOSITIONS OF A MEASURE SPACE BY AN OPERATOR SEMIGROUP

HISAKICHI SUZUKI

Let $T = \{T_t\,;\, t > 0\}$ be a strongly continuous semigroup of positive linear operators on $L_1(X) = L_1(X,\mathfrak{F},\mu)$, where $(X,\mathfrak{F},\mu)$ is a σ-finite measure space. The semigroup T decomposes X into two parts C and D, called the initially conservative and initially dissipative parts, respectively. It holds that

(1) $$T_t f = 0 \text{ on } D \text{ for any } f \in L_1(X) \text{ and } t > 0,$$

and that if $E \in \mathfrak{F}$, instead of D, satisfies the property (1), then $E \subset D$, where the inclusion holds up to a set of measure zero. (Throughout this paper, inclusions and equalities of sets or functions are considered in this sense.)

The space X has another decomposition into C^* and D^* by the same semigroup T, such that

(2) $$\|T_t(f1_{D^*})\|_1 = 0 \text{ for any } f \in L_1(X) \text{ and } t > 0,$$

and that if $E \in \mathfrak{F}$, instead of D^*, satisfies the property (2), then $E \subset D^*$, where 1_{D^*} denotes the characteristic function of D^*.

For the definitions and other characterizations of those decompositions we refer the reader to [1] in the case of contraction semigroups, and in the case of more general ones, to [2] and [3], in the latter of which C^* and D^* are denoted by P and N respectively.

In this paper we establish some theorems on the inclusion relations between D and D^*, motivated by the statement in [2] that $D \subset D^*$. If T_t are contractions, that is,

$$\|T_t\|_1 \leqq 1 \text{ for any } t > 0,$$

then an inclusion relation between D and D^* holds (Theorem 1). But, in general, there is no relation between them as Theorem 3 shows.

Theorem 1. *If $T = \{T_t\,;\, t > 0\}$ is a strongly continuous semigroup of positive contraction operators, then $D^* \subset D$.*

Proof. If $T_t (t > 0)$ are contractions, then it holds that

(3) $$T_t(L_1(C^*)) \subset L_1(C^*)$$

by [3, Remark 1], where $L_1(C^*)$ denotes the subspace consisting of functions vanishing outside C^*. Now, (3) implies together with (2) that

$$T_t f = 0 \text{ on } D^* \text{ for any } f \in L_1(X) \text{ and } t > 0,$$

and hence $D^* \subset D$.

Even in the case of contractions, it does not hold that $D = D^*$ in general.

Theorem 2. *There exists a strongly continuous semigroup* $T = \{T_t ; t > 0\}$ *of positive contractions such that* $D^* \subsetneqq D$.

Proof. Let $T = \{T_t\}$ be a strongly continuous semigroup of Markovian operators, that is,

$$(4) \qquad \int_X T_t f d\mu = \int_X f d\mu \qquad \text{for any } f \in L_1(X) \text{ and } t > 0.$$

By (2) and (4) we can easily conclude that $\mu(D^*) = 0$ for Markovian operators. If in addition $\mu(D) > 0$ holds for this semigroup T, then the theorem will be proved.

Now we assume that $X = C$ for the present Markovian semigroup T, and construct, making use of T, another Markovian semigroup $\tilde{T} = \{\tilde{T}_t\}$ such that the initially dissipative part $\tilde{D}$ determined by $\tilde{T}$ has positive measure.

Step 1. *Definition of the underlying measure space* $(\tilde{X}, \tilde{\mathfrak{F}}, \tilde{\mu})$. Let $A \in \mathfrak{F}$ have positive measure, and $(A, \mathfrak{F}_A, \mu_A)$ be the measure space defined as $\mathfrak{F}_A = \{B \cap A ; B \in \mathfrak{F}\}$ and $\mu_A(B) = \mu(B \cap A)$. Now let $(X', \mathfrak{F}', \mu')$ be a measure space isomorphic to $(A, \mathfrak{F}_A, \mu_A)$ with $X \cap X' = \emptyset$. By an isomorphism of two measure spaces A and X', we mean a one-to-one mapping τ from A onto X' such that

$$\tau^{-1}\mathfrak{F}' \subset \mathfrak{F}_A, \text{ and } \mu_A(\tau^{-1}B) = \mu'(B) \text{ for any } B \in \mathfrak{F}'.$$

We define a σ-finine measure space $(\tilde{X}, \tilde{\mathfrak{F}}, \tilde{\mu})$ as

$$\tilde{X} = X \cup X',$$
$$\tilde{\mathfrak{F}} = \{B \cup B' ; B \in \mathfrak{F} \text{ and } B' \in \mathfrak{F}'\}$$

and for $\tilde{B} = B \cup B' \in \tilde{\mathfrak{F}}$

$$\tilde{\mu}(\tilde{B}) = \mu(B) + \mu'(B').$$

Step 2. *Construction of Markovian operators* $\tilde{T}_t$ *on* $L_1(\tilde{X})$. Let $\tilde{f}$ be in $L_1(\tilde{X})$, and define a corresponding f in $L_1(X)$ as

$$(5) \qquad f(x) = \begin{cases} \tilde{f}(x) + \tilde{f}(\tau x) & \text{if } x \in A \\ \tilde{f}(x) & \text{if } x \in X \backslash A, \end{cases}$$

where $\tilde{f}$ may be any one of versions of the equivalence class.

Now we define $\tilde{T} = \{\tilde{T}_t\,;\, t > 0\}$ as

$$(\tilde{T}_t\tilde{f})(x) = \begin{cases} 0 & \text{if } x \in X' \\ (T_tf)(x) & \text{if } x \in X = \tilde{X}\backslash X' \end{cases}$$

It is shown as follows that $\tilde{T}$ forms a semigroup. Let $s, t > 0$ and $\tilde{f} \in L_1(\tilde{X})$. Now let f and h correspond respectively to $\tilde{f}$ and $\tilde{h} = \tilde{T}_t\tilde{f}$ as in (5). Then since $h = \tilde{h} = T_tf$ on X, we have

$$\tilde{T}_s(\tilde{T}_t\tilde{f}) = T_sh = T_s(T_tf) = T_{s+t}f \text{ on } X.$$

Hence $\tilde{T}_s(\tilde{T}_t\tilde{f}) = \tilde{T}_{s+t}\tilde{f}$ holds.

It is easily shown that $\tilde{T}_t$ are Markovians, and that the initially dissipative part $\tilde{D}$ contains X', which has positive measure.

Without the assumption that T is a semigroup of contractions, neither inclusion $D \subset D^*$ nor $D^* \subset D$ holds in general.

Theorem 3. *There exists a strongly continuous semigroup of positive operators which satisfies*

(a) $$\mu(D) = 0 \text{ and } \mu(D^*) > 0\,;$$

and also there exists a strongly cotinuous semigroup of positive operators which satisfies

(b) $$\mu(D) > 0,\ \mu(D^*) > 0 \text{ and } \mu(D \cap D^*) = 0.$$

Proof. Let X be the closed unit interval $[0, 1]$, and μ the Lebesgue measure on X. We construct two semigroups $T = \{T_t\,;\, t \geqq 0\}$ on $L_1(X)$.

1) For $f \in L_1(X)$ and $t \geqq 0$ we define T_t as

$$(T_tf)(x) = \begin{cases} e^{at}f(x) & \text{if } x \in [0, 1/2) \\ e^{at}f(x-1/2) & \text{if } x \in [1/2, 1], \end{cases}$$

where a is a nonzero constant. Then $T = \{T_t\,;\, t \geqq 0\}$ is a semigroup of positive operators with the norms

$$\|T_t\|_1 \leqq 2e^{at} \quad (t \geqq 0).$$

We have $\mu(D) = 0$, while $\mu(D^*) \geqq 1/2$, since it holds that for any $E \subset [1/2, 1]$ and $t \geqq 0$

$$0 \leqq \|T_t(1_E)\|_1 \leqq \|T_t(1_{[1/2,\,1]})\|_1 = 0.$$

This semigroup T satisfies the property (a).

2) Next we define T_t as

$$(T_t f)(x) = \begin{cases} 0 & \text{if } x \in [0, 1/3) \\ e^{\alpha t} f\{(x-1/3) + f(x+1/3)\} & \text{if } x \in [1/3, 2/3) \\ e^{\alpha t} f\{(x) + f(x-2/3)\} & \text{if } x \in [2/3, 1] \end{cases}$$

Clearly it holds that $\|T_t\|_1 \leqq 2e^{\alpha t}$ $(t \geqq 0)$, $D = [0, 1/3)$ and $D^* = [1/3, 2/3)$. Hence a semigroup with the property (b) is constructed.

REFERENCES

[1] M.A. AKCOGLU and R.V. CHACON: A local ratio theorem, Canad. J. Math. **22** (1970), 545—552.

[2] M.A. AKCOGLU and U. KRENGEL: A differentiation theorem in L_p, Math. Z. **169** (1979), 31—40.

[3] R. SATO: On local ergodic theorems for positive semigroups, Studia Math. **63** (1978), 45—55.

CHIBA INSTITUTE OF TECHNOLOGY

(*Received June 7, 1982*)