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# SOME COMMUTATIVITY PROPERTIES FOR RINGS

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Let $A$ be a non-empty subset of the ring $R$ with center $C$; let $N$ denote the set of nilpotent elements of $R$, and $V_R(A)$ the centralizer of $A$ in $R$. Let $q$ be a fixed integer greater than 1. We consider the following seven properties:

(I-$A$) For each $x \in R$, there exists a polynomial $f(\lambda)$ in $\boldsymbol{Z}[\lambda]$ such that $x - x^2 f(x) \in A$.

(I′-$A$) For each $x \in R$, either $x \in C$ or there exists a polynomial $f(\lambda)$ in $\boldsymbol{Z}[\lambda]$ such that $x - x^2 f(x) \in A$.

(II-$A$)$_q$ If $x, y \in R$ and $x - y \in A$, then either $x^q = y^q$ or $x$ and $y$ both belong to $V_R(A)$.

(II′-$A$)$_q$ If $x, y \in R$ and $x - y \in A$, then either $x^q = y^q$ or $[x,y] = 0$.

(II″-$A$)$_q$ If $x, y \in R$ and $x - y \in A$, then either $x^q = y^q$ or $[xy, yx] = 0$.

(III-$A$) For every $a \in A$ and $x \in R$, $[[a,x],x] = 0$.

(IV-$A$) If $a \in A$, $x \in R$ and $[a,x]^2 = 0$, then $[a,x] \in C$.

Needless to say, (I-$A$) implies (I′-$A$), (II-$A$)$_q$ does (II′-$A$)$_q$, and (II′-$A$)$_q$ does (II″-$A$)$_q$. The major purpose of this paper is to prove the following

**Theorem 1.** *The following statements are equivalent:*

1) *$R$ is commutative.*

2) *There exists a (multiplicatively) commutative subset $A$ for which $R$ satisfies* (I-$A$), (II-$A$)$_q$ *and* (III-$A$).

3) *There exists a commutative subset $A$ for which $R$ satisfies* (I-$A$), (II-$A$)$_q$ *and* (IV-$A$).

4) *There exists a commutative subset $A$ of $N$ for which $R$ satisfies* (I′-$A$) *and* (II″-$A$)$_2$.

5) *There exists a commutative subset $A$ of $N$ for which $R$ satisfies* (I′-$A$) *and* (III-$A$).

6) *There exists a commutative subset $A$ of $N$ such that $R$ satisfies* (I′-$A$) *and* (IV-$A$).

In preparation for proving Theorem 1, we establish the following two lemmas.

**Lemma 1.** (1) *Let $\psi$ be a ring homomorphism of $R$ onto $R^*$. If $R$ satisfies* (I-$A$), (I′-$A$), (II-$A$)$_q$, (II′-$A$)$_q$, (II″-$A$)$_q$ *or* (III-$A$), *then $R^*$ satisfies* (I-$\psi(A)$), (I′-$\psi(A)$), (II-$\psi(A)$)$_q$, (II′-$\psi(A)$)$_q$, (II″-$\psi(A)$)$_q$ *or* (III-$\psi(A)$), *respectively.*

(2) *If $R$ satisfies* (I′-$A$), *then $N$ is contained in $A^+ + C$, where $A^+$ is the additive subsemigroup generated by $A$.*

(3) *If $R$ satisfies* (II-$A$)$_q$, *then $[a,x^q] = 0$ for all $a \in A$ and $x \in R$.*

(4) *If $R$ satisfies* (I′-$A$) *and* (II″-$A$)$_q$ (*resp.* (I′-$A$) *and* (III-$A$)), *then $R$ is normal, that is, every idempotent $e$ of $R$ is central.*

(5) *If $A$ is commutative and $R$ satisfies* (I′-$A$), *then $N$ is a commutative nil ideal of $R$ containing the commutator ideal of $R$ and is contained in $V_R(A)$.*

(6) *If $A$ is commutative and $R$ satisfies* (I′-$A$), *then* (IV-$A$) *implies* (III-$A$), *and* (III-$A$) *does* (II″-$A$)$_q$.

*Proof.* (1) Straightforward.

(2) By a trivial induction on nilpotency index.

(3) Suppose that $[a,x] \neq 0$ for some $a \in A$ and $x \in R$. Since $(x+a)^q = x^q$ by (II-$A$)$_q$, we have $[a,x^q] = [x+a,x^q] = [x+a,(x+a)^q] = 0$.

(4) Given $x \in R$, we set $a = xe - exe$. Since $a^2 = 0$, (I′-$A$) shows that $a \in C \cup A$. Then, by (II″-$A$)$_q$ (resp. (III-$A$)), $e + a = (e+a)^q = e^q = e$ or $a = [(e+a)e, e(e+a)] = 0$ (resp. $a = [[a,e],e] = 0$). Thus in any case $a = 0$, and hence $xe = exe$. Similarly, we can show that $ex = exe$, and therefore $xe = ex$.

(5) By (2) and a theorem of Chacron (see, e.g. [5, Theorem 1]).

(6) By (5), $N$ is a commutative ideal containing the commutator ideal of $R$ and is contained in $V_R(A)$. Hence, in case (IV-$A$) is satisfied, for any $a \in A$ and $x \in R$ we have $[a,x]^2 = [a,x[a,x]] - x[a,[a,x]] = 0$. Then, $[a,x]$ is central by (IV-$A$), and therefore $R$ satisfies (III-$A$). On the other hand, if (III-$A$) is satisfied then $[x(x-a),(x-a)x] = [[a,x],x^2] = 0$ by (5) and (III-$A$). This means that if $x - y \in A$ then $[xy,yx] = 0$, and $R$ satisfies (II″-$A$)$_q$.

**Lemma 2.** *Let $R$ be a normal, subdirectly irreducible ring. If $A$ is a commutative subset of $N$ not contained in $C$ for which $R$ satisfies* (I′-$A$), *then $R$ is of characteristic $p^\alpha$, where $p$ is a prime and $\alpha > 0$.*

*Proof.* Choose $a \in A$ and $b \in R$ with $[a,b] \neq 0$. By (I′-$A$), $b - b^2 f(b) \in A$ for some $f(\lambda) \in \mathbf{Z}[\lambda]$, and hence $b^m = b^{2m} b_0^m$ with some $m > 0$ and some $b_0$ in the subring $\langle b \rangle$ generated by $b$. Since $b \notin N$ by Lemma 1 (5),

$e=(bb_0)^m$ is a non-zero central idempotent, and therefore $e$ is the identity element 1 of the subdirectly irreducible ring $R$ and $b$ is invertible. Moreover, $b^{-1}$ is integral over $\boldsymbol{Z}\cdot 1$. Since $a$ cannot commute with both $2b^{-1}$ and $3b^{-1}$, there exists an integer $k>1$ such that $[a,kb^{-1}]\neq 0$. Then, by the above argument, $(kb^{-1})^{-1}$ is integral over $\boldsymbol{Z}\cdot 1$, and hence $k^{-1}\cdot 1=(kb^{-1})^{-1}b^{-1}$ also is integral over $\boldsymbol{Z}\cdot 1$. Obviously, this implies that the additive order of 1 is non-zero, and therefore the subdirectly irreducible ring $R$ is of characteristic $p^\alpha$, where $p$ is a prime and $\alpha>0$.

We are now in a position to complete the proof of Theorem 1.

*Proof of Theorem* 1. Obviously, 1) $\Rightarrow$ 3) and 6). By Lemma 1 (6), 3) $\Rightarrow$ 2) and 6) $\Rightarrow$ 5) $\Rightarrow$ 4).

2) $\Rightarrow$ 1). In view of Lemma 1 (1), we may (and shall) assume that $R$ is subdirectly irreducible. According to [3, Theorem 19] and (I-$A$), it suffices to show that $A\subseteq C$. Suppose, to the contrary, that there exist $a\in A$ and $b\in R$ such that $[a,b]\neq 0$. By (I-$A$) and (II-$A)_q$, $b^q=(b^2f(b))^q$ with some $f(\lambda)\in\boldsymbol{Z}[\lambda]$. Since $b\notin N$ by Lemma 1 (2), Lemma 1 (4) shows that $e=(bf(b))^q$ is a non-zero central idempotent, and hence $e$ is the identity element 1 of the subdirectly irreducible ring $R$. By (I-$A$), $2-2^2g(2)\in A$ with some $g(\lambda)\in\boldsymbol{Z}[\lambda]$. Thus, we can find a non-zero integer $k$ such that $k=k\cdot 1\in A$. Obviously, $[a,b+ik]\neq 0$ for all $i\in\boldsymbol{Z}$. Hence, by (II-$A)_q$, every $b+ik$ is a zero of the polynomial $(\lambda+k)^q-\lambda^q$. Note here that $R/N$ is a subdirect sum of commutative integral domains (Lemma 1 (5)). Then, since $b+ik$ $(i=0,1,\cdots,q)$ are zeros of $(\lambda+k)^q-\lambda^q$, we can easily see that $q!k^q\in N$, and so $h\cdot 1=0$ for some positive integer $h$. This means that the characteristic of the subdirectly irreducible ring $R$ is $p^\alpha$, where $p$ is a prime and $\alpha>0$. We set $q=p^\beta t$, where $(p,t)=1$. Noting here that every non-zero idempotent of $\bar{R}=R/N$ coincides with $\bar{1}$ (Lemma 1 (4)), we can easily see that $\langle\bar{b}\rangle=\mathrm{GF}(p^\gamma)$ with some $\gamma>0$, and therefore $b^{q^\gamma}-b^{t^\gamma}\in N\subseteq V_R(A)$ (Lemma 1 (5)). Combining this with $[a,b^{q^\gamma}]=0$ (Lemma 1 (3)), we have $[a,b^{t^\gamma}]=[a,b^{q^\gamma}]-[a,b^{q^\gamma}-b^{t^\gamma}]=0$. Now, by (III-$A$), $t^\gamma b^{t^\gamma-1}[a,b]=[a,b^{t^\gamma}]=0$. Then the usual argument of replacing $b$ by $b+1$, etc. shows that $t^\gamma[a,b]=0$. Since $p^\alpha[a,b]=0$ and $(p,t)=1$, it forces a contradiction $[a,b]=0$.

4) $\Rightarrow$ 1). First, we claim that $[x^2,[x,a]]=0$ for all $a\in A$ and $x\in R$. In fact, by (II$''$-$A)_2$ and Lemma 1 (5), either

$$[x^2,[x,a]]=[(x+a)x,x(x+a)]=0,$$

or $(x+a)^2=x^2$ and hence

$$[x^2,[x,a]] = [x,[x^2,a]] = [x,[x^2,x+a]] = [x,[(x+a)^2,x+a]] = 0.$$

In view of Lemma 1 (1), we may (and shall) assume that $R$ is subdirectly irreducible. According to [3, Theorem 19] and (I'-$A$), it suffices to show that $A \subseteq C$. Suppose, to the contrary, that there exist $a \in A$ and $b \in R$ such that $[a,b] \neq 0$. Then, by Lemma 1 (4) and Lemma 2 (and its proof), $R$ is of characteristic $p^\alpha$ ($p$ a prime and $\alpha > 0$), and $\bar{b} = b+N$ is algebraic over GF($p$) (Lemma 1 (5)). Furthermore, noting that every non-zero idempotent of $R/N$ coincides with $\bar{1}$ (Lemma 1 (4)), we can easily see that $\langle \bar{b} \rangle = \mathrm{GF}(p^\beta)$ with some $\beta > 0$, and therefore $b^{p^\gamma} - b \in N$ for some $\gamma \geq \alpha$. Now, by the opening claim,

$$2[b,[b,a]] = [(b+1)^2,[b+1,a]] - [b^2,[b,a]] = 0.$$

We claim further that $[b,[b,a]] = 0$. Since $R$ is of characteristic $p^\alpha$, it suffices to consider the case $p = 2$. Then $[b^{2^\gamma} - b,[b,a]] = 0$ by Lemma 1 (5). On the other hand, $[b^2,[b,a]] = 0$ implies $[b^{2^\gamma},[b,a]] = 0$, and therefore $[b,[b,a]] = 0$ holds always. Combining the last claim with $[b^{p^\gamma} - b,a] = 0$ (Lemma 1 (5)), we obtain

$$[b,a] = [b^{p^\gamma},a] - [b^{p^\gamma} - b,a] = p^\gamma b^{p^\gamma - 1}[b,a] = 0$$

This contradiction proves that $R$ is commutative.

**Corollary 1.** *The following statements are equivalent* :

1) *$R$ is commutative.*

2) *There exists a (multiplicatively) commutative additive subsemigroup $A$ for which $R$ satisfies* (I-$A$), (II'-$A$)$_q$ *and* (III-$A$).

3) *There exists a commutative additive subsemigroup $A$ for which $R$ satisfies* (I-$A$), (II'-$A$)$_q$ *and* (IV-$A$).

*Proof.* It suffices to show that, in case $R$ satisfies (II'-$A$)$_q$ for a commutative additive subsemigroup $A$, $x - y \in A$ and $x^q \neq y^q$ imply $x \in V_R(A)$. Suppose, to the contrary, that there exists $a \in A$ such that $[a,x] \neq 0$. Then $(x+a) - x \in A$, $(x+a) - y \in A$ and $[x,y] = 0$. Since $[x+a,x] \neq 0$ and $[x+a,y] \neq 0$, we get $x^q = (x+a)^q = y^q$, a contradiction.

**Corollary 2** (cf. [7, Theorem 1] and [4, Corollary 1]). (1) *If there exists a commutative subset $A$ for which $R$ satisfies* (I-$A$) *and* (II-$A$)$_2$, *then $R$ is commutative.*

(2) *If there exists a commutative additive subsemigroup $A$ for which $R$ satisfies* (I-$A$) *and* (II'-$A$)$_2$, *then $R$ is commutative.*

*Proof.* (1) In view of Lemma 1 (1), we may assume that $R$ is subdirectly irreducible. Suppose that $[a,b] \neq 0$ for some $a \in A$ and $b \in R$. Then, as was shown in the proof of Theorem 1, $R$ has 1. Since $(b+a)^2 = b^2$ and $(b+1+a)^2 = (b+1)^2$ by $(\text{II}\cdot A)_2$, we have

$$2a = \{b^2+2(b+a)+1\}-(b+1)^2 = (b+1+a)^2-(b+1)^2 = 0,$$

and therefore $[[a,b],b] = [a,b^2]+2b^2a-2bab = 0$ by Lemma 1 (3). Thus, we have seen that $R$ satisfies (III-$A$), and $R$ is commutative by Theorem 1.

(2) This is immediate by (1) and the proof of Corollary 1.

**Corollary 3** (cf. [2, Theorem 2]). *Suppose that there exists a commutative subset $A$ of $N$ for which $R$ satisfies* (I-$A$) *and* (III-$A$). *Then $R$ is commutative.*

**Remark 1.** Theorem 1 is no longer valid if we remove the hypothesis (III-$A$) in 2) (resp. (IV-$A$) in 3)). A counterexample is given in [6, Remark, p. 18].

**Remark 2.** In [1, Theorem 1], the second author and Abu-Khuzam considered the following property:

(I*-$N$) for each $x \in R$, either $x \in C$ or there exists an integer $n > 1$ such that $x - x^n \in N$,

and proved that if $N$ is commutative and $R$ satisfies (I*-$N$) and (III-$N$), then $R$ is commutative. Also, in [8, Theorem 1], the second author considered the following property:

(II″-$A$) if $x, y \in R$ and $x-y \in A$, then either $[xy,yx] = 0$ or $x$ and $y$ both belong to $V_R(A)$,

and proved that if there exists a commutative subset $A$ of $N$ for which $R$ satisfies (I-$A$) and (II″-$A$), then $R$ is commutative. We claim here that (II″-$A$) may be restated as follows: if $x-y \in A$ then $[xy,yx] = 0$. In fact, if $x-y \in A$ and $y \in V_R(A)$, then $[x,y] = [x-y,y] = 0$, and hence $[xy,yx] = 0$. Needless to say, Theorem 1 includes both [1, Theorem 1] and [8, Theorem 1].

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