

## SKEW GROUP RINGS WITH KRULL DIMENSION

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Let  $R$  be a ring with identity, and  $G$  a group. For a given group homomorphism  $s$  from  $G$  to  $\text{Aut}(R)$ , the group of automorphisms of the ring  $R$ , the *skew group ring*  $R_sG$  is defined to be  $R_sG = \bigoplus_{g \in G} Rg$  with addition given componentwise and multiplication given as follows: if  $a, b \in R$  and  $g, h \in G$  then  $(ag)(bh) = ab^{s(g)}gh$ , where  $b^{s(g)}$  is the image of  $b$  under  $s(g)$ . In particular when  $s$  is trivial, skew group rings are just (ordinary) group rings. But, in general, since there is an interaction between the coefficient ring  $R$  and the group  $G$ , the usual techniques for group rings fail for skew group rings. However, in [5], an Artinian skew group ring has been completely characterized by the method of reducing the coefficient ring. Since Artinian rings are precisely rings with Krull dimension 0, it is quite natural to consider skew group rings with Krull dimension.

In this paper, we partially generalize the result on Artinian skew group rings in [5] to skew group rings with Krull dimension by the same technique as in [5]. Furthermore, we consider the condition for the group  $G$  whose skew group ring to have Krull dimension. When the group ring  $R[G]$  is right Noetherian, I.G. Connell [2, Theorem 2 (b)] has shown that  $G$  satisfies A.C.C. (ascending chain condition) on subgroups. Moreover, in case  $R[G]$  has Krull dimension, S.M. Woods [11, Theorem 2.2] has shown that  $G$  satisfies A.C.C. on finite subgroups. Not as nice as the last result of Connell, we obtain another condition for  $G$  whose skew group ring to have Krull dimension:  $G$  satisfies A.C.C. on normal subgroups. Also, we get the greatest lower bound of Krull dimension of the skew group ring  $R_sG$  when  $R$  is Noetherian and  $G$  is polycyclic by finite.

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Let  $M$  be a right  $R$ -module. The Krull dimension of  $M$  which will be denoted by  $\text{Kdim } M$ , is defined by transfinite recursion as follows: if  $M = 0$  then  $\text{Kdim } M = -1$ ; if  $\alpha$  is an ordinal and  $\text{Kdim } M \not\prec \alpha$  then  $\text{Kdim } M = \alpha$ , provided there is no infinite descending chain  $M_0 \supseteq M_1 \supseteq M_2 \supseteq \dots$  of submodules such that  $\text{Kdim } (M_{i-1}/M_i) \not\prec \alpha$  ( $i = 1, 2, \dots$ ). It is possible that there is no ordinal  $\alpha$  such that  $\text{Kdim } M = \alpha$ . In that case we say that  $M$

has no Krull dimension. The Krull dimension of a ring,  $\text{Kdim } R$ , is defined to be the Krull dimension of the right  $R$ -module  $R$ . We note that the modules of Krull dimension 0 are precisely non-zero Artinian modules. It has been shown that every Noetherian module has Krull dimension (see [3, Proposition 1.3]).

By reducing the coefficient ring as in [5], we get the following which corresponds to the result of [11, Theorem 2.2].

**Proposition 1.** *If a skew group ring  $R_sG$  has Krull dimension, then  $G$  satisfies A.C.C. on finite subgroups.*

*Proof.* Let  $J$  denote the Jacobson radical of  $R$ . Then, since  $J$  is invariant under  $s(G)$ , the skew group ring of  $G$  over  $R/J$  can be defined. Also we note that this skew group ring has Krull dimension. So we may (and shall) assume that the ring  $R$  is semiprimitive with Krull dimension. By [3, Corollary 3.4]  $R$  has a right classical quotient ring  $Q$  which is semiprimitive Artinian. Moreover, since every ring automorphism of  $R$  can be extended naturally to  $Q$ , the skew group ring  $Q_sG$  makes sense. In this case it is easily verified that the map  $\sigma$  from the lattice of right ideals of  $Q_sG$  to that of  $R_sG$  defined by  $\sigma(I) = I \cap R_sG$  is strictly increasing. Hence we have  $\text{Kdim } Q_sG \leq \text{Kdim } R_sG$ .

Now decompose  $Q$  into simple components:  $Q = Q_1 \oplus Q_2 \oplus \cdots \oplus Q_n$ . Let  $H = \{g \in G \mid Q_i^{s(g)} = Q_i \text{ for } i = 1, 2, \dots, n\}$ . Then  $H$  is a normal subgroup of  $G$  with finite index. By [5, Proposition 2.1],  $Q_{1,s}H$  has Krull dimension. Let  $F$  be the center of  $Q_1$ . Then  $F$  is a field invariant under the action of  $s(H)$ . Hence  $s$  induces a group homomorphism  $u$  from  $H$  to  $\text{Aut}(F)$ . By [5, Lemma 2.3],  $F_uH$ , and hence  $F^{u(H)}[H]$ , has Krull dimension. Hence, by [11, Theorem 2.2],  $H$  satisfies A.C.C. on finite subgroups. Since  $G/H$  is finite,  $G$  satisfies A.C.C. on finite subgroups. This completes the proof.

The following corollary is a partial generalization of [5, Theorem 3.3].

**Corollary 2.** *Let  $G$  be a locally finite group. Then a skew group ring  $R_sG$  has Krull dimension  $\alpha$  if and only if  $G$  is finite and  $R$  has Krull dimension  $\alpha$ .*

Proposition 1 and Corollary 2 are not true for crossed products in general. D.S. Passman [7, Proposition 4.2] has constructed an Artinian twisted group ring of an infinite locally finite group over an Artinian ring.

By the same fashion as we proved Proposition 1 and by [11, Theorem 2.4], we get the following

**Corollary 3.** *If  $G$  is abelian and a skew group ring  $R_sG$  has Krull dimension, then  $G$  is finitely generated.*

The following examples show that the converses of Proposition 1 and Corollary 2 do not hold even when  $G$  acts faithfully on the coefficient ring.

**Example 4.** Let  $S$  be the  $2 \times 2$  matrix ring over the real number field  $\mathbf{R}$ . For each  $a$  in  $\mathbf{R}$ , the map  $g_a$  on  $S$  defined by  $g_a(A) = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} A \begin{pmatrix} 1 & -a \\ 0 & 1 \end{pmatrix}$  is a ring automorphism. Under the composition of maps, the set  $\{g_a \mid a \in \mathbf{R}\}$  forms an abelian group. Take a transcendental number  $b$ . Then  $\{g_{b^k} \mid k \geq 1\}$  generates an infinitely generated free abelian group satisfying A.C.C. on finite subgroups, but the skew group ring has no Krull dimension by Corollary 3.

**Example 5.** Let  $K$  be a finite field, and  $F$  an algebraic closure of  $K$ . Then  $\text{Aut}_K(F)$  is abelian and every element except  $1_F$  has infinite order. Now, let  $\sigma$  be an element of  $\text{Aut}_K(F)$  different from  $1_F$ . Then, as we shall see later (Proposition 13), the skew group ring of  $\langle \sigma \rangle$  over  $F$  has Krull dimension 1 but the group  $\langle \sigma \rangle$  is not locally finite.

Next, we consider A.C.C. on normal subgroups when skew group rings have Krull dimension. The following lemma is due to wholly to P.F. Smith.

**Lemma 6.** *Let  $F$  be a field, and  $G$  a group. If the group ring  $F[G]$  has Krull dimension, then  $G$  satisfies A.C.C. on normal subgroups.*

*Proof.* Let  $\text{Kdim } F[G] = \alpha$ . We proceed by induction on  $\alpha$ . If  $\alpha = 0$ , then  $F[G]$  is Artinian and so  $G$  is finite by [2, Theorem 1]. Now suppose  $\alpha > 0$ . According to [11, Theorem 2.2], without loss of generality, we may assume that  $G$  contains no finite normal subgroups different from 1. Thus  $F[G]$  is a prime Goldie ring by [2, Theorem 8] and [3, Corollary 3.4]. Let  $1 \subseteq N_1 \subseteq N_2 \subseteq \dots$  be an ascending chain of normal subgroups of  $G$ . Then the left annihilator of  $\omega N_1 = \sum_{g \in N_1} (g-1)F[G]$  is zero and hence  $\omega N_1$  contains a non-divisor of zero, and so  $\text{Kdim } F[G/N_1] < \text{Kdim } F[G]$  (see [8, Lemma 1.2]). By induction method, we have  $N_t = N_{t+1} = \dots$  for some  $t \geq 1$ .

By using the same method as in the proof of Proposition 1, we get the following from Lemma 6.

**Proposition 7.** *If a skew group ring  $R_sG$  has Krull dimension, then  $G$  satisfies A.C.C. on normal subgroups.*

Let  $\chi$  be a class of groups. A group  $G$  is a hyper- $\chi$  group if every non-trivial homomorphic image of  $G$  contains a non-trivial normal subgroup which is a  $\chi$  group.

By Proposition 7 and [6, Theorem 10.2.7] we have the following

**Corollary 8.** *Let  $G$  be a hyper-(abelian or finite) group, and  $R$  a right Noetherian ring. Then the following are equivalent :*

- (1)  $R_sG$  has Krull dimension.
- (2)  $G$  is polycyclic by finite.
- (3)  $G$  satisfies A.C.C. on subgroups.

In [2, Theorem 2 (b)], I.G. Connell has shown that if  $R[G]$  is right Noetherian then  $G$  satisfies A.C.C. on subgroups. Proposition 7 is not as nice as Connell's. But fortunately, P.F. Smith has brought to the author the following

**Proposition 9.** *Let  $K$  be a field, and  $G$  a linear group. Then  $K[G]$  has Krull dimension if and only if  $G$  is polycyclic by finite.*

*Proof.* The sufficiency follows from [6, Theorem 10.2.7]. In what follows we prove the necessity. Assume that  $K[G]$  has Krull dimension. We claim that  $G$  contains no non-cyclic free subgroups. Let  $F$  be a free subgroup of  $G$  if such exists. Then  $K[F]$  has Krull dimension, and so  $K[F]$  is an Ore domain. But  $K[F]$  is a free ideal ring, and so a principal right ideal domain by [1, Proposition 2.2, p.48]. Thus  $F$  is cyclic. By Tit's theorem [10, Corollary 10.17],  $G$  contains a solvable normal subgroup  $S$  such that  $G/S$  is locally finite. By making use of Corollary 3 and the technique employed in the proof of [8, Theorem 5.16], we can see that  $K[S]$  having Krull dimension implies that  $S$  is polycyclic, and also  $K[G/S]$  having Krull dimension implies that  $G/S$  is finite (see [11, Corollary 2.3]).

Now, we consider a bound on the Krull dimensions of crossed products of polycyclic by finite groups over Noetherian rings. Before doing this, we show by two examples that the skew group ring  $R_sG$  may not have Krull dimension even if either  $R$  has Krull dimension and  $G$  is infinite cyclic, or

$R$  is simple Artinian and  $G$  is locally finite. Note that  $G$  acts non-trivially on  $R$  in the following examples.

**Example 10.** Let  $R = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \mid a \in \mathbf{Z}, b \in \mathbf{Z}_p \right\}$ . Then  $R$  is not Noetherian but has Krull dimension ([3, Example 10.6]). Define a ring automorphism  $\sigma$  of  $R$  by  $\sigma \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} = \begin{pmatrix} a & -b \\ 0 & a \end{pmatrix}$  and a group homomorphism  $s$  from the infinite cyclic group  $G = \langle \sigma \rangle$  to  $\text{Aut}(R)$  by  $s(g^m) = \sigma^m$ . In this case, the skew group ring  $R_s G$  has no Krull dimension. For, if  $R_s G$  has Krull dimension then  $R_s H = R[H]$  has Krull dimension, where  $H = \text{Ker } s$ . Since the group  $H$  is infinite cyclic,  $R$  is right Noetherian by [11, Theorem 3.1], a contradiction.

**Example 11.** Let  $K$  be an infinite field of prime characteristic  $p$ . And let  $R = \text{Mat}_2(K)$ . For each  $a \in K$ , define an automorphism  $g_a$  as in Example 4. Then  $G = \{g_a \mid a \in K\}$  is a locally finite group. By Corollary 2, the skew group ring of  $G$  over  $R$  has no Krull dimension even if  $R$  is simple Artinian.

**Lemma 12.** *Let  $R$  be right Noetherian and let a group  $G$  be polycyclic by finite with Hirsch number  $h(G)$ . Then every crossed product  $R * G$  is right Noetherian and  $\text{Kdim } R * G \leq h(G) + \text{Kdim } R$ .*

*Proof.* See [4, Theorem 18] and [6, Theorem 10.2.7].

In [9], an example shows that there exists a right Noetherian ring  $R$  such that  $\text{Kdim } R_s G < 1 + \text{Kdim } R$  for infinite cyclic  $G$ .

We describe in detail the Krull dimensions of skew group rings of polycyclic by finite groups over Artinian rings. In [8, Theorem 2.5] it has been shown that  $\text{Kdim } R[G] = h(G) + \text{Kdim } R$  when  $R$  is right Noetherian and  $G$  is polycyclic by finite with Hirsch number  $h(G)$ . Also, as we mentioned earlier, an example shows that this type of results does not hold for skew group rings. To calculate the Krull dimension of the skew group ring over an Artinian ring, we note that the Krull dimension of the group ring  $K[G]$  of a polycyclic by finite group over a field  $K$  is  $h(G)$  (see [8, Corollary 2.6]).

**Proposition 13.** *If  $R$  is right Noetherian and  $G$  is polycyclic by finite with Hirsch number  $h(G)$ , then  $\text{Kdim } R_s G \geq h(G)$ . In particular, if  $R$  is right Artinian then  $\text{Kdim } R_s G = h(G)$ .*

*Proof.* As in the proof of Proposition 1, we may assume that  $R$  is semiprimitive. By Goldie's theorem  $R$  has a right classical quotient ring  $Q$  which is semiprimitive Artinian. We note that  $\text{Kdim } Q_s G \leq \text{Kdim } R_s G$ . Decompose  $Q$  into simple components:  $Q = Q_1 \oplus Q_2 \oplus \cdots \oplus Q_n$ . Let  $N = \{g \in G \mid Q_i^{s(g)} = Q_i \text{ for } i = 1, 2, \dots, n\}$ . Then  $N$  is a normal subgroup of  $G$  with finite index, and so we have  $h(G) = h(N)$ . Since  $Q_1$  is invariant under  $N$ , we can define a group homomorphism  $u$  from  $N$  to  $\text{Aut}(Q_1)$  by  $u(g) =$  the restriction of  $s(g)$  to  $Q_1$ . Then, as in the last part of the proof of Proposition 1, we get  $\text{Kdim } R_s G \geq \text{Kdim } Q_s G \geq \text{Kdim } Q_s N \geq \text{Kdim } Q_{1,u} N \geq \text{Kdim } F^{u(N)}[N] = h(N) = h(G)$ . Also, by Lemma 12, we have  $\text{Kdim } R_s G = h(G)$  for right Artinian  $R$ . This completes the proof.

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