Math. J. Okayama Univ. 25 (1983), 69—73

ON DUALITY IN Γ-RINGS

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Morita contexts and Γ-rings are equivalent concepts. From a Morita context, we obtain a Γ-ring through the triple products defined in Jacobson [1, p. 166]. On the other hand, a Γ-ring gives rise to a Morita context (see Kyuno [2]). Therefore, the duality theory obtained in Morita contexts can be interpreted in terms of Γ-rings. However, it may be of some interest to derive the duality theory in Γ-rings directly, which we shall do in this note. First, we define a Γ-ring of homomorphisms between two additive groups. Then, a Γ-ring-module will be defined. It consists of two ring-modules which are connected by operations of the Γ-ring. Under the assumption of existence of unities, these two ring-modules correspond each other in a unique way, which is the duality. A general Γ-ring case will be discussed lastly. Since there are some different definitions for a Γ-ring, the Γ-ring described in this note will be given a new name. We call it a context, which consists of two modules and two triple products satisfying some associative laws. The classical example of contexts is a system consisting of a ring-module and its dual. From it, we can obtain one of Morita duality theorems.

We begin with the definition of Γ-rings of homomorphisms. Let M and N be additive groups, and let S and T be subgroups of Hom(M, N) and of Hom(N, M) respectively. When $STS \subseteq S$ and $TST \subseteq T$, we say that (S, T) is a *Γ-ring of homomorphisms* between M and N, where for example ST indicates $\{\sum s_i t_i \mid s_i \in S \text{ and } t_i \in T\}$. Throughout this note, (S, T) stands for a Γ-ring of homomorphisms. In this case, TS is a subring of Hom(M, M) and ST is one of Hom(N, N).

Definition. Let A be an TS-module and B an ST-module. We say that (A, B) is an *(S, T)-module* if there exist homomorphisms $S \to \bar{S} \subseteq$ Hom (A, B) (of additive groups A and B) and $T \to \bar{T} \subseteq$ Hom(B, A), where the bar indicates the mappings, such that $\bar{t}\bar{s}a = (ts)a$, $\overline{s_1 t s_2}a = \bar{s}_1\bar{t}\bar{s}_2 a$, $\bar{s}\bar{t}b = (st)b$ and $\overline{t_1 s t_2}b = \bar{t}_1\bar{s}\bar{t}_2 b$. We assume that if TS (or ST) contains the unity then A (or B) is a unitary module. In this case, we simply denote $\bar{s}a$ by sa, etc. For example, (TS, S) is an (S, T)-module.

Let (A, B) and (C, D) be (S, T)-modules. If $C \subseteq A$, then clearly $D \subseteq B$. In this case, we say that (C, D) is an *(S, T)-submodule* of (A, B).

C is clearly an TS-submodule of A. Conversely we have

Lemma 1. *Let (A, B) be an (S, T)-module. If C is an TS-submodule of A, then there exists an ST-module D such that (C, D) is an (S, T)-submodule of (A, B).*

Proof. Let $D = SC =$ the subgroup generated by sc $(s \in S, c \in C)$. $(ST)D = (ST)SC \subseteq SC = D$. D is an ST-module. It is easy to see that (C, D) is an (S, T)-module.

A factor module of an (S, T)-module is defined in a natural way. If (C, D) is an (S, T)-submodule of (A, B), then $(A/C, B/D)$ is considered to be an (S, T)-module, which we call a factor module.

Proposition 1. *For every TS-module A with $TSA = A$, there exists an ST-module B such that (A, B) is an (S, T)-module.*

Proof. Let F be a free TS-module with a basis $\{A\}$. Every element of F is uniquely expressed as $\sum r_i\{a_i\}$ $(r_i \in TS, \{a_i\} \in \{A\})$. On the other hand, define H as a set of all expressions $\sum s_i\{a_i\}$ $(s_i \in S)$. We can make H an additive group. It is also easy to conclude that (F, H) is an (S, T)-module. It is a direct sum of (TS, S). Let K be the kernel of the homomorphism f of F to A defined through $f(\{a\}) = a$ for $a \in A$. By Lemma 1, there exists an ST-module L such that (K, L) is an (S, T)-submodule of (F, H). Now consider the factor module $(F/K, H/L)$. Identify F/K with A and let $B = H/L$.

Proposition 2. *Let (A, B) be an (S, T)-module. If ST contains the unity, then $B = SA$.*

Proof. If ST contains the unity, then $B = STB \subseteq SA$. $B \supseteq SA$ is trivial. So, $B = SA$.

Definition. Let (A, B) and (C, D) be (S, T)-modules. If there exist homomorphisms (of additive groups) $f: A \to C$ and $g: B \to D$ such that $sf(a) = g(sa)$ and $tg(b) = f(tb)$, we say that (f, g) is an *(S, T)-homomorphism* of (A, B) to (C, D).

If (f, g) is an (S, T)-homomorphism of (A, B) to (C, D), then f is an TS-homomorphism of A to C and g an ST-homomorphism of B to D.

Lemma 2. *Suppose that (f, g) is an (S, T)-homomorphism of (A, B)*

to (C, D). *If* ST *contains the unity then* g *is uniquely determined by* f.

Proof. If ST contains the unity, $B = SA$ by Proposition 2. Every element of B is $b = \sum s_i a_i$. Then, $g(b) = \sum s_i f(a_i)$.

We denote the correspondence $f \to g$ in Lemma 2 by $\varphi(A, C)$.

Proposition 3. *Suppose that* ST *contains the unity. If* (A, B) *and* (C, D) *are* (S, T)*-modules, then there exists a homomorphism* $\varphi(A, C)$: $\mathrm{Hom}_{TS}(A, C) \to \mathrm{Hom}_{ST}(B, D)$. *If* (E, F) *is an* (S, T)*-module, then* $\varphi(C, E)\,\varphi(A, C) = \varphi(A, E)$.

Proof. By the assumption, every element of B is $b = \sum s_i a_i$. We define a mapping g of B to D by $g(b) = \sum s_i f(a_i)$ for an element f in $\mathrm{Hom}_{TS}(A,C)$. To show that g is well defined, we must show that $\sum s_i' a_i = 0$ implies $\sum s_i' f(a_i) = 0$. Let $1 = \sum s_j t_j$ be the unity in ST. Then $\sum s_i' a_i = 0$ implies

$$0 = \sum s_j f(t_j \sum s_i' a_i) = \sum s_j t_j \sum s_i' f(a_i) = \sum s_i' f(a_i).$$

It is easy to see that (f, g) is an (S, T)-homomorphism of (A, B) to (C, D). By Lemma 2, we have a mapping $\varphi(A, C) : f \to g$. The latter is almost clear.

Summarizing all the propositions, we obtain the duality theorem.

Theorem. *Suppose that* (S, T) *is a* Γ*-ring of homomorphisms such that* ST *and* TS *contain the unities. Then, between all* TS*-modules and* ST*-modules, there exists a unique one to one correspondence given by* $A \leftrightarrow B$ *where* (A, B) *is an* (S, T)*-module. When* $A \leftrightarrow B$ *and* $C \leftrightarrow D$, *we have an isomorphism* $\varphi(A, C) : \mathrm{Hom}_{TS}(A, C) \to \mathrm{Hom}_{ST}(B, D)$, *which satisfies* $\varphi(C, E)\varphi(A, C) = \varphi(A, E)$ *for any* TS*-module* E.

Categorically the above theorem implies that the correspondence $A \leftrightarrow B$ between TS-modules and ST-modules gives equivalent functors between the category of all TS-modules and that of all ST-modules.

Definition. A *context* (S, T, τ, μ) consists of additive groups S and T and of trilinear mappings τ and μ such that $\tau : S \otimes T \otimes S \to S$ and μ: $T \otimes S \otimes T \to T$, satisfying $s_1 t_1 (s_2 t_2 s_3) = (s_1 t_1 s_2) t_2 s_3 = s_1 (t_1 s_2 t_2) s_3$, where we denote $\tau(s, t, s')$ by sts' and $\mu(t, s, t')$ by tst'.

Let (S, T, τ, μ) be a context. We define a bilinear mapping h of $T \otimes S$ to $\mathrm{Hom}(S, S)$ by $h(t, s)(s') = s'ts$. Denote $h(T \otimes S)$ by TS. Then an element s in S induces a homomorphism $\bar{s}$ of TS to S by $\bar{s}(h(t_1, s_1)) =$

$h(t_1, s_1)(s) = st_1s_1$. Let $\bar{S} = \{\bar{s} \mid s \in S\}$. $S \to \bar{S}$ is a homomorphism and the kernel consists of all s such that $st_1s_1 = 0$ for all t_1 and s_1. On the other hand, an element t in T induces a homomorphism $\bar{t}$ of S to TS by $\bar{t}(s) = h(t, s)$. Let $\bar{T} = \{\bar{t} \mid t \in T\}$. $T \to \bar{T}$ is a homomorphism, whose kernel consists of all t such that $s'ts = 0$ for all s and s' in S. Practically we could assume that the kernel $= 0$ and $T = \bar{T}$, for we could replace T by $\bar{T}$ in the beginning.

Proposition 4. *$(\bar{S}, \bar{T})$ is a Γ-ring of homomorphisms between $M = TS$ and $N = S$.*

Proof. We must only check that $\bar{s}_1\bar{t}\bar{s}_2 \in \bar{S}$ and $\bar{t}_1\bar{s}\bar{t}_2 \in \bar{T}$. $\bar{s}_1\bar{t}\bar{s}_2(t's) = \bar{s}_1\bar{t}(s_2t's) = \bar{s}_1(h(t, s_2t's)) = s_1t(s_2t's) = (s_1ts_2)t's = \overline{s_1ts_2}(t's)$. Therefore, $\bar{s}_1\bar{t}\bar{s}_2 = \overline{s_1ts_2} \in \bar{S}$. Also, $\bar{t}_1\bar{s}\bar{t}_2(s') = \bar{t}_1\bar{s}(h(t_2, s')) = \bar{t}_1(st_2s') = h(t_1, st_2s')$. Then, $[\bar{t}_1\bar{s}\bar{t}_2(s')](s'') = s''t_1(st_2s') = s''t_1(st_2s') = s''ts'$, where $t = t_1st_2 \in T$. But, $s''ts' = h(t, s')(s'') = \bar{t}(s')(s'')$. Therefore, $\bar{t}_1\bar{s}\bar{t}_2 = \bar{t} \in \bar{T}$.

Thus, for a context (S, T, τ, μ), if $\bar{S}\bar{T}$ and $\bar{T}\bar{S}$ contain the unities, we have the duality theorem. The following is a classical example.

Example. Let S be an R-module, where R is a ring with unity. Let $T = \mathrm{Hom}_R(S, R)(= S^*$, the dual of $S)$. If $t \in T$, then $t(s) \in R$ for $s \in S$. We define τ and μ as follows:

$$\tau : S \otimes T \otimes S \to S \text{ by } \tau(s_1, t, s_2) = t(s_2)s_1.$$
$$\mu : T \otimes S \otimes T \to T \text{ by } \mu(t_1, s, t_2)(s') = t_2(s')t_1(s).$$

As before, we denote $\tau(s_1, t, s_2) = s_1ts_2$ and $\mu(t_1, s, t_2) = t_1st_2$. So, $s_1ts_2 = t(s_2)s_1$ $(t(s_2) \in R)$, and $t_1st_2(s') = t_2(s')t_1(s) \in R$. Now, we check the associativity laws. $s_1t_1(s_2t_2s_3) = s_1t_1[t_2(s_3)s_2] = t_1[t_2(s_3)s_2]s_1 = t_2(s_3)t_1(s_2)s_1$, since t_1 is an R-homomorphism. $(s_1t_1s_2)t_2s_3 = [t_1(s_2)s_1]t_2s_3 = t_2(s_3)t_1(s_2)s_1$. Also, $s_1(t_1s_2t_2)s_3 = [t_1s_2t_2(s_3)]s_1 = t_2(s_3)t_1(s_2)s_1$. Therefore, the associativity laws hold and (S, T, τ, μ) is a context. Lastly, we investigate the conditions for the existence of the unities of $\bar{T}\bar{S}$ ahd $\bar{S}\bar{T}$. $M = TS = h(T \otimes S)$ and $h(t, s)(s') = s'ts = t(s)s'$ by definition, where $t(s) \in R$. We identify $h(t, s)$ with $t(s)$ and hence $M = h(T \otimes S) \subseteq R$. For $h(t_1, s_1) \in M$ and $\bar{s} \in \bar{S}$, $\bar{s}h(t_1, s_1) = h(t_1, s_1)s = t_1(s_1)s$. If $\bar{t} \in \bar{T}$, $\bar{t}\bar{s} = h(t, s) = t(s)$, and hence $\bar{t}\bar{s}h(t_1s_1) = t(t_1(s_1)s) = t_1(s_1)t(s) = h(t_1, s_1)t(s)$. In other words, the homomorphism $\bar{t}\bar{s}$ is the right multiplication of the element $t(s)$. Therefore, if there exist t_i and s_i such that $1 = \sum t_i(s_i)$, then $1 = \sum \bar{t}_i\bar{s}_i$(the unity in $\bar{T}\bar{S}$).

Especially, if $M = h(T \otimes S) = R$, then $\overline{T}\overline{S}$ contains the unity. Note that this condition is satisfied if S is a generator ([1, p. 173]). Similarly, we have $\bar{s}\bar{t}s_1 = t(s_1)s$, which indicates that the homomorphism $\bar{s}\bar{t}$ is given by $s_1 \rightarrow t(s_1)s$. Thus, $\overline{S}\overline{T}$ can contain the unity if there exist t_i and s_i such that $\sum t_i(s)s_i = s$ for any s in S. Note that this condition is satisfied if S is a finitely generated projective module ([1, p. 153]).

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(*Received December 2, 1982*)