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ON THE CARTAN INVARIANTS OF p-SOLVABLE GROUPS

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Throughout the present paper, k will represent an algebraically closed field of characteristic $p > 0$. Let G be a finite p-solvable group, and B a block ideal of defect d of the group algebra kG. In [3], Fong proved that each Cartan invariant of B is always bounded above by p^d. On the other hand, Koshitani [6] proved that the nilpotency index of the Jacobson radical of B is bounded above by p^d, that is, the Loewy length of each projective indecomposable B-module is not greater than p^d. In this paper, we consider the possibility that the composition length of each projective indecomposable B-module is not greater than p^d. In other words, we consider the possibility that

(∗) each row-sum of the Cartan matrix of B is bounded above by p^d.

In §1, we consider the case that G has p-length 1, and prove that the Cartan matrix of every block ideal of kG has property (∗) if and only if $G/O_{p'p}(G)$ is abelian. Furthermore, we prove that if every irreducible B-module has k-dimension a power of p, then the Cartan matrix of B has property (∗). Now, let G be an arbitrary finite group, and H a normal subgroup of G. Let B and b be block ideals of kG and kH, respectively, such that B covers b. In §2 (resp. §3), we consider the case that $[G:H] = p$ (resp. $[G:H] = q$, a prime number different from p), and the relationship between the Cartan invariants of B and those of b will be investigated. As a consequence, we show that if $[G:H]$ is a power of p and the Cartan matrix of b has property (∗), then the Cartan matrix of B also has property (∗). However, in general, the converse need not be true; a counterexample will be given in §4.

Throughout this paper, all modules are assumed to be finitely generated right modules. We denote by $P_G(M)$ the projective cover of a kG-module M. If H is a subgroup of G, then $M|_H$ is a kH-module obtained from M by restricting the domain of operators to kH. Given a kH-module L, we denote by L^G the induced module $L \otimes_{kH} kG$. The Jacobson radical of kG is denoted by J_G. Given a block ideal B of kG, we denote by C_B and $\delta(B)$ the Cartan matrix of B and a defect group of B, respectively.

1. Let G be a p-solvable group, and B an arbitrary block ideal of

kG. In [9], Schwarz proved that if $G/O_{p'p}(G)$ is abelian then each row-sum of C_B is equal to $|\delta(B)|$. Furthermore, the converse of this fact has been proved in [8]. First, by making use of these results, we prove the following

Theorem 1. *Let G be a p-solvable group of order $p^a m$ $(a \geqq 1,\ p \nmid m)$. If G has p-length 1, then the following are equivalent:*

(1) *$G/O_{p'p}(G)$ is abelian.*

(2) *If B is an arbitrary block ideal of kG, then each row-sum of C_B is bounded above by $|\delta(B)|$.*

(3) *If B is an arbitrary block ideal of kG, then each row-sum of C_B is equal to $|\delta(B)|$.*

(4) *If B_0 is the principal block ideal of kG, then each row-sum of C_{B_0} is bounded above by p^a.*

(5) *If B_0 is the principal block ideal of kG, then each row-sum of C_{B_0} is equal to p^a.*

Proof. In view of [9, Satz 6.3] and [8, Theorem 5], it suffices to show that (4) implies (1).

Suppose that (4) holds. Since G has p-length 1, $G/O_{p'}(G)$ has a normal Sylow p-subgroup. As is well known, B_0 is isomorphic to $kG/O_{p'}(G)$. Hence, we may assume that $O_{p'}(G)=1$ and G has a normal Sylow p-subgroup. Then kG itself is the principal block ideal of kG. Now, let $\{F_1, F_2, \cdots, F_s\}$ be a full set of non-isomorphic irreducible kG-modules, where F_1 is a trivial kG-module. We put $f_i = \dim_k F_i$ and $u_i = \dim_k P_G(F_i)$ $(1 \leqq i \leqq s)$. Then p does not divide f_i because G has a normal Sylow p-subgroup. As is well known, $P_G(F_i)$ is isomorphic to a direct summand of $F_i \otimes_k P_G(F_1)$. Hence, by [4, Theorem 2B], we have $u_i = p^a f_i$ for all i. Now, we may assume that f_s is a maximal one among f_i's, and that $f_t = f_{t+1} = \cdots = f_s$. Suppose that $f_s > 1$, and so $t > 1$. Let c_{ij} be the (i,j)-entry of the Cartan matrix C of kG (the multiplicity of F_j as a composition factor of $P_G(F_i)$). Then, by our assumption, there holds that

$$p^a f_l = u_l = \textstyle\sum_{i=1}^{s} c_{li} f_i \leqq (\sum_{i=1}^{s} c_{li}) f_l \leqq p^a f_l \quad (t \leqq l \leqq s).$$

This implies that if $c_{li} \neq 0$ then $f_i = f_l$. Hence we have $c_{ij} = 0$ provided $t \leqq i \leqq s$ and $1 \leqq j \leqq t-1$. However, this is impossible, because C is indecomposable. Hence $f_s = 1$. This implies that every irreducible kG-module has k-dimension 1, and hence $G/O_p(G)$ is abelian, proving (1).

Next, we prove the following

Proposition 2. *Let G be a p-solvable group, and B a block ideal of kG. If every irreducible B-module has k-dimension a power of p, then each row-sum of C_B is bounded above by $|\delta(B)|$.*

Proof. Let $\{F_1, F_2, \cdots, F_s\}$ be a full set of non-isomorphic irreducible B-modules. Then, by assumption, $\dim_k P_G(F_i) = p^a$ for all i, where p^a is the order of a Sylow p-subgroup of G ([4, Theorem 2B]). Now, we put $\dim_k F_i = p^{e_i}$ $(1 \leqq i \leqq s)$. We may assume that e_1 is minimal among e_i's. Let c_{ij} be the (i,j)-entry of C_B. Then we have

$$p^a = \dim_k P_G(F_i) = \textstyle\sum_{j=1}^{s} c_{ij} \dim_k F_j \geqq (\sum_{j=1}^{s} c_{ij}) p^{e_1}.$$

Since $|\delta(B)| = p^{a-e_1}$, the above implies that

$$|\delta(B)| = p^a/p^{e_1} \geqq \textstyle\sum_{j=1}^{s} c_{ij} \quad \text{for all } i,$$

proving the assertion.

2. Let H be a normal subgroup of a finite group G, and b a block ideal of kH. We denote by $T_G(b)$ the inertial subgroup of b:

$$T_G(b) = \{g \in G \mid g^{-1} f g = f\},$$

where f is a central primitive idempotent of kH such that $b = fkH$. Given an irreducible b-module L, we denote by $T_G(L)$ the inertial subgroup of L:

$$T_G(L) = \{g \in G \mid L \otimes_{kH} g \cong L \text{ as } kH\text{-modules}\}.$$

One may remark that $T_G(L)$ is contained in $T_G(b)$. Now, let $\{g_i \mid 1 \leqq i \leqq t\}$ be a right transversal of $T_G(b)$ in G. Then $e = \sum_{i=1}^{t} g_i^{-1} f g_i$ is a central idempotent of kG. If $e = e_1 + e_2 + \cdots + e_m$ is the decomposition of e into (orthogonal) central primitive idempotents of kG, then we say that each block ideal $e_i kG$ covers b.

Throughout the subsequent study in this section, we suppose that $[G:H] = p$. Our objective is to find some relationship between Cartan invariants of b and those of a block ideal of kG which covers b. We notice here that if L is an irreducible kH-module then $T_G(L)$ is either H or G.

At first, we prove the following

Lemma 3. *Let L be an irreducible kH-module. Then there holds the following:*

(1) *If $T_G(L) = H$, then L^G is an irreducible kG-module and $P_G(L^G) \cong P_H(L)^G$.*

(2) *If $T_G(L) = G$, then there exists a unique (up to isomorphism) irreducible kG-module W such that $W|_H \cong L$; and then $P_G(W) \cong P_H(L)^G$.*

Proof. (1) It is well known that L^G is an irreducible kG-module ([2, Chap. III, (2.11)]). Since $P_H(L)^G$ is a projective kG-module and

$$P_H(L)^G/(P_H(L)J_H)^G \cong (P_H(L)/P_H(L)J_H)^G \cong L^G,$$

we see that $P_G(L^G) \cong P_H(L)^G$.

(2) It is well known that there exists a unique irreducible kG-module W such that $W|_H \cong L$ ([2, Chap. III, (3.16)]) and that $P_H(L)^G$ is a projective indecomposable kG-module ([2, Chap. III, (3.13)]). Since W is isomorphic to an irreducible submodule of L^G and L^G is isomorphic to a submodule of $P_H(L)^G$, W is isomorphic to the socle of $P_H(L)^G$. This implies that $P_G(W) \cong P_H(L)^G$.

Now, let L_1, L_2, V_1 and V_2 be irreducible kH-modules such that $T_G(L_i) = H$ and $T_G(V_i) = G$ $(i = 1, 2)$. We put $M_i = L_i^G$. Further, we denote by W_i an irreducible kG-module such that $W_i|_H \cong V_i$. Let σ be an element of G such that $\{1, \sigma, \cdots, \sigma^{p-1}\}$ is a right transversal of H in G. Given a k-space X and a positive integer n, we denote by nX a direct sum of n copies of X. Then, in virtue of Frobenius reciprocity theorem and the preceding lemma, we can easily see the next

Lemma 4. (1) $\mathrm{Hom}_{kG}(P_G(M_1), P_G(M_2)) \cong$
$\oplus_{i=0}^{p-1} \mathrm{Hom}_{kH}(P_H(L_1), P_H(L_2)\otimes_{kH}\sigma^i) \cong \oplus_{i=0}^{p-1} \mathrm{Hom}_{kH}(P_H(L_1)\otimes_{kH}\sigma^i, P_H(L_2))$.

(2) $\mathrm{Hom}_{kG}(P_G(M_1), P_G(W_1)) \cong p\mathrm{Hom}_{kH}(P_H(L_1), P_H(V_1))$
$\cong \oplus_{i=0}^{p-1} \mathrm{Hom}_{kH}(P_H(L_1)\otimes_{kH}\sigma^i, P_H(V_1))$.

(3) $\mathrm{Hom}_{kG}(P_G(W_1), P_G(M_1)) \cong \oplus_{i=0}^{p-1} \mathrm{Hom}_{kH}(P_H(V_1), P_H(L_1)\otimes_{kH}\sigma^i)$
$\cong p\mathrm{Hom}_{kH}(P_H(V_1), P_H(L_1))$.

(4) $\mathrm{Hom}_{kG}(P_G(W_1), P_G(W_2)) \cong p\mathrm{Hom}_{kH}(P_H(V_1), P_H(V_2))$.

By [1, Theorem 54.16], we see that *if X and Y are irreducible kG-modules, then* $\dim_k\mathrm{Hom}_{kG}(P_G(X), P_G(Y))$ *is equal to the multiplicity of X as a composition factor of $P_G(Y)$.* Hence, the above gives a linkage between the Cartan invariants of kG and those of kH.

The next can be proved by [2, Chap.V, (3.5)], [5, Proposition 4.2] and [7, Theorem 6.11].

Lemma 5. *If b is a block ideal of kH, then b is covered by a unique block ideal B of kG, and there holds the following*:

(1) *If $T_G(b) = H$ then $\delta(B) \underset{G}{=} \delta(b)$.*

(2) *If $T_G(b) = G$ then $\delta(B) \cap H \underset{G}{=} \delta(b)$ and $|\delta(B)| = p|\delta(b)|$.*

Now, let b be a block ideal of kH such that $T_G(b) = H$. Then $T_G(L)$

$= H$ for every irreducible b-module L. Let B be a block ideal of kG which covers b, and M an irreducible B-module. Then $M|_H$ is a completely reducible kH-module by Clifford's theorem, and there exists a composition factor L of $M|_H$ belonging to b. Since L^G is an irreducible kG-module (Lemma 3 (1)) and $\mathrm{Hom}_{kG}(L^G, M) \cong \mathrm{Hom}_{kH}(L, M|_H) \neq 0$, we have $L^G \cong M$. Now, let $\{L_1, L_2, \cdots, L_s\}$ be a full set of non-isomorphic irreducible b-modules. We put $b_i = \sigma^{-(i-1)} b \sigma^{i-1}$, $L_{ji} = L_j \otimes_{kH} \sigma^{i-1}$ and $M_j = L_j^G$ $(1 \leqq i \leqq p;\ 1 \leqq j \leqq s)$. Then, Lemmas 4 and 5 together with the above fact imply the following result which is a special case of [2, Chap. V, (2.5)].

Proposition 6. (1) *$\{M_1, M_2, \cdots, M_s\}$ is a full set of non-isomorphic irreducible B-modules, and $\{L_{1i}, L_{2i}, \cdots, L_{si}\}$ is a full set of non-isomorphic irreducible b_i-modules $(1 \leqq i \leqq p)$.*

(2) *B and b have the same Cartan matrix, and have a defect group in common.*

Next, suppose that b is a block ideal of kH with $T_G(b) = G$. Then the inertial subgroup $T_G(L)$ of any irreducible b-module L is either H or G. Let B be a block ideal of kG which covers b. If M is an irreducible B-module, then there exists a composition factor L of a completely reducible kH-module $M|_H$ belonging to b. If $T_G(L) = H$ then, as stated just before Proposition 6, L^G is isomorphic to M. On the other hand, if $T_G(L) = G$ then $M|_H \cong L$ by Lemma 3 (2). Now, let $\{L_{11}, \cdots, L_{1p};\ \cdots;\ L_{r1}, \cdots, L_{rp};\ V_1, V_2, \cdots, V_t\}$ be a full set of non-isomorphic irreducible b-modules, where $T_G(L_{i1}) = H$, $L_{ij} = L_{i1} \otimes_{kH} \sigma^{j-1}$ $(1 \leqq i \leqq r;\ 1 \leqq j \leqq p)$ and $T_G(V_l) = G$ $(1 \leqq l \leqq t)$. Put $M_i = L_{i1}^G$ $(1 \leqq i \leqq r)$, and choose an irreducible kG-module W_l such that $W_l|_H \cong V_l$ $(1 \leqq l \leqq t)$. Then $\{M_1, \cdots, M_r;\ W_1, \cdots, W_t\}$ is a full set of non-isomorpnic irreducible B-modules. Given irreducible kG-modules X, Y (resp. irreducible kH-modules A, B), we denote by c_{XY} (resp. $\tilde{c}_{AB}$) the multiplicity of Y (resp. B) as a composition factor of $P_G(X)$ (resp. $P_H(A)$). Then, by Lemma 4, we have the following

Proposition 7. (1) $c_{M_iM_j} = \sum_{l=1}^{p} \tilde{c}_{L_{i1}L_{jl}} = \sum_{l=1}^{p} \tilde{c}_{L_{i2}L_{jl}} = \cdots = \sum_{l=1}^{p} \tilde{c}_{L_{ip}L_{jl}}$ $(1 \leqq i, j \leqq r)$.

(2) $c_{W_iM_j} = \sum_{l=1}^{p} \tilde{c}_{V_iL_{jl}}$ $(1 \leqq i \leqq t;\ 1 \leqq j \leqq r)$.

(3) $c_{M_iW_j} = p\tilde{c}_{L_{i1}V_j} = p\tilde{c}_{L_{i2}V_j} = \cdots = p\tilde{c}_{L_{ip}V_j}$ $(1 \leqq i \leqq r;\ 1 \leqq j \leqq t)$.

(4) $c_{W_iW_j} = p\tilde{c}_{V_iV_j}$ $(1 \leqq i, j \leqq t)$.

We are now in a position to state the following

Theorem 8. *Let N be a normal subgroup of G such that G/N is a p-group. Let b be a block ideal of kN. If B is a block ideal of kG which covers b, then there holds the following*:

(1) *If each row-sum of C_b is bounded above by $|\delta(b)|$, then each row-sum of C_B is bounded above by $|\delta(B)|$.*

(2) *The converse of* (1) *is true, provided one of the following conditions holds*:

(i) $T_G(b) = H$.

(ii) *$T_G(L) = G$ for every irreducible b-module L.*

Proof. (1) By Propositions 6, 7 and Lemma 5 (2).

(2) If (i) holds, then the converse is true by Proposition 6. On the other hand, if (ii) holds then $C_B = [G:N]C_b$ by Proposition 7. Since $|\delta(B)| = [G:N]\,|\delta(b)|$ (Lemma 5 (2)), the converse of (1) is also true.

The next is a combination of Theorems 1 and 8.

Corollary 9. *Let G be a group such that $G = O_{p'pp'p}(G)$ and $O_{p'pp'}(G)/O_{p'p}(G)$ is abelian, and let B be a block ideal of kG. Then each row-sum of C_B is bounded above by $|\delta(B)|$.*

3. Throughout this section, we assume that H is a normal subgroup of G with $[G:H] = q$, a prime different from p. We notice here that if L is an irreducible kH-module and $T_G(L) \neq H$ then $T_G(L) = G$. We establish first three lemmas which correspond to Lemmas 3, 4 and 5, respectively.

Lemma 10. *Let L be an irreducible kH-module. Then there holds the following*:

(1) *If $T_G(L) = H$, then L^G is an irreducible kG-module and $P_G(L^G) \cong P_H(L)^G$.*

(2) *If $T_G(L) = G$, then there exist q non-isomorphic irreducible kG-modules $W_1, W_2, \cdots, W_q$ such that $W_i|_H \cong L$; and then $P_G(W_i)|_H \cong P_H(L)$.*

Proof. (1) It is well known that L^G is an irreducible kG-module ([2, Chap. III, (2.11)]). It is also clear that $P_H(L)^G$ is a projective kG-module. Noting that $J_G = J_H kG$, we get

$$P_H(L)^G/P_H(L)^G J_G \cong (P_H(L)/P_H(L)J_H)^G \cong L^G.$$

Hence, $P_H(L)^G \cong P_G(L^G)$.

(2) The existence of such W_i's is well known ([10, Lemma 1]). Observing $J_G = J_H kG$, we get

$$P_G(W_i)|_H \cong P_H(P_G(W_i)|_H) \cong P_H(P_G(W_i)|_H/(P_G(W_i)|_H)J_H) \cong P_H(P_G(W_i)|_H/(P_G(W_i)J_G)|_H) \cong P_H(W_i|_H) \cong P_H(L).$$

Let L_1, L_2, V_1 and V_2 be irreducible kH-modules such that $T_G(L_i) = H$ and $T_G(V_i) = G$ $(i = 1, 2)$. We put $M_i = L_i^G$, and choose an irreducible kG-module W_1 such that $W_1|_H \cong V_1$. Let τ be an element of G such that $\{1, \tau, \cdots, \tau^{q-1}\}$ is a right transversal of H in G.

Lemma 11. (1) $\mathrm{Hom}_{kG}(P_G(M_1), P_G(M_2)) \cong \bigoplus_{i=0}^{q-1} \mathrm{Hom}_{kH}(P_H(L_1), P_H(L_2)\otimes_{kH}\tau^i) \cong \bigoplus_{i=0}^{q-1} \mathrm{Hom}_{kH}(P_H(L_1)\otimes_{kH}\tau^i, P_H(L_2))$.

(2) $\mathrm{Hom}_{kG}(P_G(M_1), P_G(W_1)) \cong \mathrm{Hom}_{kH}(P_H(L_1), P_H(V_1))$.

(3) $\mathrm{Hom}_{kG}(P_G(W_1), P_G(M_1)) \cong \mathrm{Hom}_{kH}(P_H(V_1), P_H(L_1))$.

(4) *If $W_{11}, \cdots, W_{1q}$ (resp. $W_{21}, \cdots, W_{2q}$) are non-isomorphic irreducible kG-modules such that $W_{1i}|_H \cong V_1$ (resp. $W_{2i}|_H \cong V_2$), and if $1 \leq l \leq q$, then*

$$\sum_{i=1}^{q} \dim_k \mathrm{Hom}_{kG}(P_G(W_{1i}), P_G(W_{2l})) = \dim_k \mathrm{Hom}_{kH}(P_H(V_1), P_H(V_2)).$$

Proof. (1), (2) and (3) are clear by Frobenius reciprocity theorem and Lemma 10.

(4) Observing that $J_G = J_H kG$, we get

$$P_H(V_2)J_H^r/P_H(V_2)J_H^{r+1} \cong (P_G(W_{2l})|_H)J_H^r/(P_G(W_{2l})|_H)J_H^{r+1} \cong (P_G(W_{2l})J_G^r/P_G(W_{2l})J_G^{r+1})|_H,$$

where r is an arbitrary non-negative integer and $1 \leqq l \leqq q$. This shows that the multiplicity of V_1 as a composition factor of $P_H(V_2)J_H^r/P_H(V_2)J_H^{r+1}$ coincides with that of W_{1i} as a composition factor of $P_G(W_{2l})J_G^r/P_G(W_{2l})J_G^{r+1}$ $(1 \leqq i \leqq q)$. Since $\dim_k\mathrm{Hom}_{kH}(P_H(V_1), P_H(V_2))$ (resp. $\dim_k\mathrm{Hom}_{kG}(P_G(W_{1i}), P_G(W_{2l}))$) is equal to the multiplicity of V_1 (resp. W_{1i}) as a composition factor of $P_H(V_2)$ (resp. $P_G(W_{2l})$), the assertion (4) follows immediately.

The next is obvious by [5, Proposition 4.2].

Lemma 12. *Let B and b be block ideals of kG and kH, respectively. If B covers b, then B and b have a defect group in common.*

Now, we consider the case that the inertial subgroup $T_G(b)$ of a block ideal b of kH coincides with H.

Lemma 13. *Let b be a block ideal of kH. If $T_G(b) = H$ then b is covered by a uniquely determined block ideal of kG.*

Proof. Suppose that more than one block ideal of kG covers b, and let $B_1, B_2, \cdots, B_m$ be all such block ideals of kG. By an argument similar to that employed in the paragraph preceding Proposition 6, we see that if M is an irreducible B-module then there exists an irreducible b-module L such that $M \cong L^G$. So, we let $L_1, L_2, \cdots, L_m$ be irreducible b-modules such that L_i^G belongs to B_i $(1 \leqq i \leqq m)$. Now, if L and L' are irreducible b-modules such that L^G and L'^G belong to different block ideals of kG, then $\mathrm{Hom}_{kG}(P_G(L^G), P_G(L'^G)) = 0$, and so $\mathrm{Hom}_{kH}(P_H(L), P_H(L')) = 0$ by Lemma 11 (1). Thus, we see that $\tilde{c}_{L_iL_1} = 0$ for $i \neq 1$, where $\tilde{c}_{L_iL_1}$ is the multiplicity of L_1 as a composition factor of $P_H(L_i)$. But this is impossible, because C_b is indecomposable. Hence b is covered by a uniquely determined block ideal of kG.

Let b be a block ideal of kH such that $T_G(b) = H$, and B a block ideal of kG which covers b. Let $\{L_1, L_2, \cdots, L_s\}$ be a full set of non-isomorphic irreducible b-modules. Now, putting $b_i = \tau^{-(i-1)} b \tau^{i-1}$, $L_{ji} = L_j \otimes_{kH} \tau^{i-1}$ and $M_j = L_j^G$ $(1 \leqq i \leqq q\,;\, 1 \leqq j \leqq s)$, by Lemmas 10—13 we get the following which is a special case of [2, Chap. V, (2.5)].

Proposition 14. (1) *$\{M_1, M_2, \cdots, M_s\}$ is a full set of non-isomorphic irreducible B-modules, and $\{L_{1i}, L_{2i}, \cdots, L_{si}\}$ is a full set of non-isomorphic irreducible b_i-modules $(1 \leqq i \leqq q)$.*

(2) *B and b have the same Cartan matrix and have a defect group in common.*

Next, suppose that b is a block ideal of kH with $T_G(b) = G$. Then, for any irreducible b-module L, $T_G(L)$ is either H or G. Let $\{B_1, B_2, \cdots, B_m\}$ be a full set of block ideals of kG covering b. We put $B = B_1 \oplus B_2 \oplus \cdots \oplus B_m$. If M is an irreducible B-module, then there exists a composition factor L of a completely reducible kH-module $M|_H$ belonging to b. If $T_G(L) = H$ then, as in the paragraph preceding Proposition 6, we see that $M \cong L^G$; and if $T_G(L) = G$ then $M|_H \cong L$ by Lemma 10 (2). Now, let $\{L_{11}, \cdots, L_{1q}\,;\, \cdots\,;\, L_{r1}, \cdots, L_{rq}\,;\, V_1, V_2, \cdots, V_t\}$ be a full set of non-isomorphic irreducible b-modules, where $T_G(L_{i1}) = H$, $L_{ij} = L_{i1} \otimes_{kH} \tau^{j-1}$ $(1 \leqq i \leqq r\,;\, 1 \leqq j \leqq q)$ and $T_G(V_l) = G$ $(1 \leqq l \leqq t)$. We put $M_i = L_{i1}^G$ $(1 \leqq i \leqq r)$, and we let $\{W_{l1}, \cdots, W_{lq}\}$ be a full set of non-isomorphic irreducible kG-modules such that $W_{lj}|_H \cong V_l$ $(1 \leqq l \leqq t)$. Then $\{M_1, M_2, \cdots, M_r\,;\, W_{11}, \cdots, W_{1q}\,;\, \cdots\,;\, W_{t1}, \cdots, W_{tq}\}$ is a full set of non-isomorphic irreducible B-modules. Further, according to Lemma 11, we can prove the following proposition which corresponds to Proposition 7.

Proposition 15. (1) $c_{M_iM_j}=\sum_{l=1}^{q}\tilde{c}_{L_{i1}L_{jl}}=\sum_{l=1}^{q}\tilde{c}_{L_{i2}L_{jl}}=\cdots=\sum_{l=1}^{q}\tilde{c}_{L_{iq}L_{jl}}$ $(1\leqq i, j\leqq r)$.

(2) $c_{W_{i1}M_j}=c_{W_{i2}M_j}=\cdots=c_{W_{iq}M_j}=\tilde{c}_{V_iL_{j1}}=\tilde{c}_{V_iL_{j2}}=\cdots=\tilde{c}_{V_iL_{jq}}$ $(1\leqq i \leqq t;\ 1\leqq j\leqq r)$.

(3) $c_{M_iW_{j1}}=c_{M_iW_{j2}}=\cdots=c_{M_iW_{jq}}=\tilde{c}_{L_{i1}V_j}=\tilde{c}_{L_{i2}V_j}=\cdots=\tilde{c}_{L_{iq}V_j}$ $(1\leqq i \leqq r;\ 1\leqq j\leqq t)$.

(4) $\sum_{l=1}^{q}c_{W_{i1}W_{jl}}=\sum_{l=1}^{q}c_{W_{i2}W_{jl}}=\cdots=\sum_{l=1}^{q}c_{W_{iq}W_{jl}}=\tilde{c}_{V_iV_j}$ $(1\leqq i, j\leqq t)$.

We are now in a position to state the following

Theorem 16. *Let H be a normal subgroup of G with $[G{:}H]=q$. Let b be a block ideal of kH, and B a block ideal of kG which covers b. Then there holds the following:*

(1) *Suppose that, for every irreducible b-module, its inertial subgroup coincides with G. If each row-sum of C_b is bounded above by $|\delta(b)|$, then that of C_B is bounded above by $|\delta(B)|$.*

(2) *Suppose that, for every irreducible b-module, its inertial subgroup coincides with H. Then, B is the unique block ideal covering b, and the the following statements are equivalent:*

(i) *Each row-sum of C_b is bounded above by $|\delta(b)|$.*

(ii) *Each row-sum of C_B is bounded above by $|\delta(B)|$.*

Proof. (1) By Proposition 15.

(2) In the same way as in the proof of Lemma 13, we can see that b is covered uniquely by a block ideal of kG, even if $T_G(b)$ is different from H. The rest of the assertion follows from Propositions 14 and 15.

4. In this section, we assume $p=3$ and give a counterexample which shows that the converse of Theorem 8 (1) need not be true.

Let $U=\langle u\rangle\times\langle v\rangle$ be an elementary abelian group of order 3^2. We look upon U as a vector space over GF(3). Then SL(2,3) acts naturally on U. We denote by G a semi-direct product of U by SL(2,3) with respect to this action. We notice that $|\mathrm{SL}(2,3)|=24$ and a Sylow 2-subgroup Q of SL(2,3) is a quaternion group. We let $Q=\langle a, b \mid a^4=1,\ a^2=b^2,\ b^{-1}ab=a^{-1}\rangle$, and denote by $\langle s\rangle$ a Sylow 3-subgroup of SL(2,3). Then we may, and shall assume that $G=\langle u, v, a, b, s\rangle$ and

$$a^{-1}ua=u^2v,\ b^{-1}ub=uv,\ a^{-1}va=uv,\ b^{-1}vb=uv^2,$$
$$s^{-1}us=uv,\ s^{-1}vs=v,\ s^{-1}as=b,\ s^{-1}bs=ba.$$

In what follows, we put $X=U\langle a^2\rangle$, $Y=U\langle a\rangle$ and $H=UQ$. Now, by making use of Propositions 7 and 15, we shall determine the Cartan matrices of kH and kG.

To begin with, we shall determine the Cartan matrix of kX. Put $\varepsilon_1 = -(1+a^2)$ and $\varepsilon_2 = -1+a^2$. Then $1 = \varepsilon_1+\varepsilon_2$ is a decomposition of 1 into orthogonal primitive idempotents in kX. By a brief computation, we can see that $\{\varepsilon_i, \varepsilon_i u\varepsilon_i, \varepsilon_i v\varepsilon_i, \varepsilon_i uv\varepsilon_i, \varepsilon_i u^2 v\varepsilon_i\}$ is a k-basis of $\varepsilon_i kX\varepsilon_i$ $(i = 1, 2)$ and that $\{\varepsilon_1 u\varepsilon_2, \varepsilon_1 v\varepsilon_2, \varepsilon_1 uv\varepsilon_2, \varepsilon_1 u^2 v\varepsilon_2\}$ is a k-basis of $\varepsilon_1 kX\varepsilon_2$. Hence, we have

Lemma 17. *The Cartan matrix of kX is given by* $\begin{pmatrix} 5 & 4 \\ 4 & 5 \end{pmatrix}$.

Next, we shall determine the Cartan matrix of kY. Put $e_1 = 1+a+a^2+a^3$, $e_2 = 1-a+a^2-a^3$, $e_3 = 1+\xi a-a^2-\xi a^3$ and $e_4 = 1-\xi a-a^2+\xi a^3$, where ξ is a primitive 4-th root of 1 in k. Then $1 = e_1+e_2+e_3+e_4$ is a decomposition of 1 into orthogonal primitive idempotents in kY. Put $L_i = \varepsilon_i kX/\varepsilon_i J_X$ $(i = 1, 2)$ and $M_j = e_j kY/e_j J_Y$ $(1 \leqq j \leqq 4)$. Then it is easy to see that $T_Y(L_1) = T_Y(L_2) = Y$, $M_1|_X \cong M_2|_X \cong L_1$ and $M_3|_X \cong M_4|_X \cong L_2$. By Lemma 17 and Proposition 15, we get the following:

$$\begin{aligned} c_{M_1M_1}+c_{M_1M_2} &= c_{M_2M_1}+c_{M_2M_2} = 5, \\ c_{M_1M_3}+c_{M_1M_4} &= c_{M_2M_3}+c_{M_2M_4} = 4, \\ c_{M_3M_1}+c_{M_3M_2} &= c_{M_4M_1}+c_{M_4M_2} = 4, \\ c_{M_3M_3}+c_{M_3M_4} &= c_{M_4M_3}+c_{M_4M_4} = 5. \end{aligned}$$

On the other hand, we can see that $\{e_i, e_i ue_i, e_i ve_i\}$ is a k-basis of $e_i kYe_i$ $(i=1, 3)$ and $\{e_1 ue_3, e_1 ve_3\}$ is a k-basis of $e_1 kYe_3$. Thus, $c_{M_1M_1} = c_{M_3M_3} = 3$ and $c_{M_1M_3} = 2$. Noting here that the Cartan matrix is symmetric, we get $c_{M_iM_i} = 3$ $(1 \leqq i \leqq 4)$ and $c_{M_iM_j} = 2$ $(1 \leqq i \neq j \leqq 4)$.

Lemma 18. *The Cartan matrix of kY is given by* $\begin{pmatrix} 3 & 2 & 2 & 2 \\ 2 & 3 & 2 & 2 \\ 2 & 2 & 3 & 2 \\ 2 & 2 & 2 & 3 \end{pmatrix}$.

Now, we determine the Cartan matrix of kH. Put

$$\begin{aligned} f_1 &= -(1+a+a^2+a^3)(1+b), \\ f_2 &= -(1+a+a^2+a^3)(1-b), \\ f_3 &= -(1-a+a^2-a^3)(1+b), \\ f_4 &= -(1-a+a^2-a^3)(1-b), \\ f &= -(1-a^2). \end{aligned}$$

Noting that $kH/U \cong kQ$, we see that f_1, f_2, f_3 and f_4 are orthogonal primitive idempotents of kH and f can be decomposed into two orthogonal primitive idempotents of kH, say f_5 and f_6. Thus, $1 = f_1+f_2+f_3+f_4+f_5+f_6$ is a decomposition of 1 into orthogonal primitive idempotents in kH. Let $N_i = f_ikH/f_iJ_H$ $(1 \leqq i \leqq 6)$. Then, it is easy to see that $T_H(M_1) = T_H(M_2) = H$, $T_H(M_3) = T_H(M_4) = Y$, $N_1|_Y \cong N_2|_Y \cong M_1$, $N_3|_Y \cong N_4|_Y \cong M_2$ and that $N_5 \cong N_6 \cong M_3^H \cong M_4^H$. Now, Proposition 15 together with Lemma 18 yields the following:

$$\begin{aligned}
&c_{N_1N_1}+c_{N_1N_2} = c_{N_2N_1}+c_{N_2N_2} = 3,\\
&c_{N_1N_3}+c_{N_1N_4} = c_{N_2N_3}+c_{N_2N_4} = 2,\\
&c_{N_3N_1}+c_{N_3N_2} = c_{N_4N_1}+c_{N_4N_2} = 2,\\
&c_{N_3N_3}+c_{N_3N_4} = c_{N_4N_3}+c_{N_4N_4} = 3,\\
&c_{N_5N_1} = c_{N_5N_2} = c_{N_5N_3} = c_{N_5N_4} = 2,\\
&c_{N_5N_5} = 5.
\end{aligned}$$

On the other hand, we can see that $\{f_i, f_iuf_i\}$ is a k-basis of f_ikHf_i $(i = 1, 3)$ and $\{f_1uf_3\}$ is a k-basis of f_1kHf_3. Hence, $c_{N_1N_1} = c_{N_3N_3} = 2$ and $c_{N_1N_3} = 1$. Now, we can find all the Cartan invariants of kH as in the next lemma.

Lemma 19. *The Cartan matrix of kH is given by* $\begin{pmatrix} 2 & 1 & 1 & 1 & 2\\ 1 & 2 & 1 & 1 & 2\\ 1 & 1 & 2 & 1 & 2\\ 1 & 1 & 1 & 2 & 2\\ 2 & 2 & 2 & 2 & 5 \end{pmatrix}$.

In conclusion, we determine the Cartan matrix of kG. It is easy to see that $T_G(N_1) = G$ and $T_G(N_2) = T_G(N_3) = T_G(N_4) = H$. Now, suppose that $T_G(N_5) = H$. Then N_5, $N_5 \otimes_{kH} s$ and $N_5 \otimes_{kH} s^2$ are non-isomorphic irreducible kH-modules. But this is impossible, because N_5 is the only one (up to isomorphism) irreducible kH-module with k-dimension 2. Hence $T_G(N_5) = G$. Thus, we see that f_1kG, f_2kG and f_5kG are non-isomorphic projective indecomposable kG-modules (Lemma 3). Putting $F_1 = f_1kG/f_1J_G$, $F_2 = f_2kG/f_2J_G$ and $F_3 = f_5kG/f_5J_G$, we see that $F_1|_H \cong N_1$, $F_3|_H \cong N_5$ and $F_2 \cong N_2^G \cong N_3^G \cong N_4{}^G$. Hence, by Lemma 19 and Proposition 7, we can get the Cartan matrix of kG.

Theorem 20. *The Cartan matrix of kG is given by* $\begin{pmatrix} 6 & 3 & 6\\ 3 & 4 & 6\\ 6 & 6 & 15 \end{pmatrix}$.

Obviously, each row-sum of the Cartan matrix of kG is not greater than 27, the order of a Sylow 3-subgroup of G. However, the 5-th row-sum of the Cartan matrix of kH exceeds 9, the order of a Sylow 3-subgroup of H.

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