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ON THE NILPOTENCY INDEX OF THE RADICAL OF A GROUP ALGEBRA. IV

Dedicated to Professor Kentaro Murata on his 60th birthday

KAORU MOTOSE

Throughout this paper, we shall use the following notations: Let p be a fixed prime number, let G be a finite group with a p-Sylow subgroup P of order p^a, let KG be a group algebra of G over a field K of characteristic p and let $t(G)$ be the nilpotency index of the radical $J(KG)$ of KG. Further, given a finite subset S of KG, $\hat{S}$ denotes the sum of all elements of S.

First, in § 1, we shall investigate $t(G)$ of a group G with a split (B, N)-pair. Next, in § 2, we shall state some remarks concerning a theorem of Morita [5]. § 3 is devoted to studying a p-solvable group G with $t(G) = a(p-1)+1$. Unfortunately, we have not characterized yet a group G of minimal order such that G is p-solvable, $t(G) = a(p-1)+1$ and P is not elementary abelian. However, we can establish the structure of such a group under certain extra conditions (Theorem 12). § 4 contains an example of a group H satisfying the following conditions:

(1) H possesses all the conclusions of Lemma 11.

(2) $t(H) > \beta(p-1)+1$ where p^β is the order of a p-Sylow subgroup of H.

Finally, by making use of results in § 3, we shall prove that there exist no finite rings satisfying certain conditions (§ 5).

1. We begin with stating a lemma which is efficient in studying the nilpotency index of the radical of a group algebra.

Lemma 1. *Let A be a ring. Let B, I and J be subsets of A satisfying the following conditions:*

(1) $IAI = IBI$, (2) $IJ = JI$, (3) $BJ \subseteq JB$.

Then $(JIA)^n \subseteq J^n IA$. Moreover, if $J^n = 0$ then JIA is contained in the radical of A.

Proof. Clearly, the result holds for $n = 1$. Assume the result for n. Then we conclude that $(JIA)^{n+1} \subseteq J^n IAJIA = J^n IAIJA = J^n IBIJA = J^n IBJIA \subseteq J^n IJBIA = J^{n+1} IBIA \subseteq J^{n+1} IA$.

Theorem 2. *Suppose that a group G has finite subgroups H and U such that $G = UN_G(H)U$ and $H \subseteq N_G(U)$. Then $(J(KH)\hat{U}KG)^{t(H)} = 0$ and $J(KH)\hat{U} \subseteq J(KG)$.*

Proof. Taking A to be KG, B to be $KN_G(H)$, I to be $\{\hat{U}\}$, and J to be $J(KH)$ in Lemma 1, we readily obtain the conclusion.

We suggest the effectiveness of Theorem 2 by giving the following example which played a fundamental role in [8].

Example 3. Let $q = p^e$, r a divisor of $q-1$, and s a multiple of p with $(s, q-1) = 1$. Let H be the Galois group of $R = GF(q^s)$ over $S = GF(q)$ and let T be $\langle b^r \rangle$ where b is a generator of the multiplicative group of R. We consider the following permutation groups on R: $V = \{v_a : x \to x+a \mid a \in R\}$ and $U = \{u_t : x \to tx \mid t \in T\}$, and put $G = \langle H, U, V \rangle$. Since $q-1$ and $(q^s-1)/(q-1)$ are relatively prime, every element of R is a product of an element of T and an element of S. This shows that $G = UC_G(H)U$ since $C_G(H)$ contains $\{v_a \mid a \in S\}$ and $u_t v_a u_t^{-1} = v_{ta}$. Noting that $hu_t h^{-1} = u_{h(t)}$ for all $h \in H$, by Theorem 2, we obtain that $J(KH)\hat{U} \subseteq J(KG)$ and the nilpotency index of $J(KH)\hat{U}KG$ is the p-part of s.

Theorem 2 is also applicable to finite groups of Lie type (see [1]).

Proposition 4. *Suppose that a finite group G has a split (B, N)-pair of characteristic r such that B is a semi-direct product of a normal r-subgroup U and an abelian r'-subgroup $H = B \cap N$ (see [1]). Then $J(KH)\hat{U} \subseteq J(KG)$.*

Proof. Since $G = UNU$ by Bruhat decomposition and H is a normal subgroup of N, the assertion follows from Theorem 2.

Although the next is stated for a special type of a finite group, similar observation is possible for finite groups of Lie type, too.

Proposition 5. *Let r be a prime number, $q = r^e$, p an odd prime divisor of $q-1$, and $G = SL(2, q)$. Then $t(G)$ is the p-part p^m of $q-1$.*

Proof. Obviously, G has a split (B, N)-pair, where B is the subgroup of upper triangular matrices and N is the subgroup of monomial matrices. Thus $H = B \cap N$ is the subgroup of diagonal matrices which is isomorphic to the multiplicative group of the finite field of q elements. Thus, by

Proposition 4, we have $p^m = t(H) \leq t(G)$. Since the order of G is $q(q-1)\cdot(q+1)$, a p-Sylow subgroup of G is a cyclic group of order p^m, and so $t(G) = p^m$ by Dade's theorem [2] (see also [7]).

2. Throughout this section, we shall use the following notations: Let N be a normal subgroup (not necessarily a p'-subgroup) of a finite group G and let e be a centrally primitive idempotent of KN. We set $H = \{x \in G \mid xex^{-1} = e\}$ and $\tilde{e} = \sum_{i=1}^{s} a_i e a_i^{-1}$, where $\{a_i\}$ $(a_1 = 1)$ is a set of representatives of the right cosets of H in G.

The following interesting theorem has been proved by K. Morita.

Theorem 6 (Morita [5]). *If K is algebraically closed and KNe is simple, then $KG\tilde{e}$ is isomorphic to a complete matrix algebra over a twisted group algebra of H/N.*

The next is immeadiate from the preceding theorem.

Corollary 7. *Assume that K is algebraically closed. If $N = H$ and KNe is simple, then $KG\tilde{e}$ is simple.*

This result applies especially to the radical of a group algebra of a Frobenius group.

Corollary 8 ([6]). *Let G be a Frobenius group with kernel N and complement W. If p divides the order of W then $J(KG) = J(KW)\hat{N}$.*

Proof. We may assume, in the usual way, that K is algebraically closed. If e is not equal to $|N|^{-1}\hat{N}$, then $H = N$ implies that $KG\tilde{e}$ is simple (see [3, the proof of (25.4)]). Thus we obtain the assertion.

As an application of Lemma 1, we have the following

Proposition 9. $J(KHe) = J(KH)e \subseteq J(KG)$.

Proof. Setting $A = KG$, $B = KH$, $I = \{e\}$ and $J = J(KH)$, we have $IAI = eKGe = \sum_{i=1}^{s} KHea_i e a_i^{-1} a_i = KHe = eKHe = IBI$. Also, it is easy to check other assumptions of Lemma 1. Thus we have $J(KH)e \subseteq J(KG)$.

The next shows that the converse of Corollary 7 holds.

Corollary 10. *If $N = H$ and $KG\tilde{e}$ is simple, then KNe is simple.*

Proof. By Proposition 9, $0=J(KG)\tilde{e}\supseteq J(KN)e\tilde{e}=J(KN)e=J(KNe)$. Thus KNe is simple.

3. Throughout this section, let G be a group of the minimal order which satisfies the following conditions:

1. G is a p-solvable group.
2. P is not elementary abelian.
3. $t(G)=a(p-1)+1$.

We show that G possesses the properties listed in the following lemma.

Lemma 11. (1) $O_{p'}(G)=1$.

(2) *$U=O_p(G)$ ($\neq 1$) is elementary abelian and $G=O_{p,p',p}(G)$.*

(3) $|G|\leq p^{p+1}(p^p-1)/(p-1)$.

(4) *$U=[U, V]$, $C_U(V)=1$ and G is a semi-direct product of U by $N_G(V)$ where V is a p'-subgroup such that $O_{p,p'}(G)=UV$.*

(5) *$V=[W, V]$ where W is a p-Sylow subgroup of $N_G(V)$.*

(6) *$N_G(V)=WV$ is isomorphic to a subgroup of $Aut(U)=GL(U)$.*

(7) *U is a minimal normal subgroup of UV.*

Proof. (1) We have $t(G)\geq t(G/O_{p'}(G))\geq a(p-1)+1$ (see [12]). Thus $O_{p'}(G)=1$ by the minimality of the order of G.

(2) From the inequality $t(G)\geq t(G/U)+t(U)-1$ (see [12]) we see that both U and P/U are elementary abelian. Hence G/U is of p-length 1, and so $H=O_{p,p',p}(G)$ is a normal subgroup of G whose index is a p'-number. Since $t(G)=t(H)$ (see [11]), we get $G=H=O_{p,p',p}(G)$.

(3) It is known that there exists a group H of order $p^{p+1}(p^p-1)/(p-1)$ such that $t(H)=p^2$ and a p-Sylow subgroup of H is nonabelian (see [8]). Hence, we have $p^{p+1}(p^p-1)/(p-1)\geq |G|$.

(4) We can see that $G=N_G(V)U$ by Frattini argument, and $N_U(V)=C_U(V)$. Thus, to prove the last assertion, it suffices to show that $C_U(V)$ is trivial. In view of $C_G(U)=U$ (see [4, Theorem 3.2, p.228]), V can be regarded as an automorphism group of U by conjugation, and consequently $U=C_U(V)\times[V, U]$ (see [4, Theorem 2.3, p.177]). Clearly, $[V, U]=[UV, U]$ is a non-trivial normal subgroup by $O_{p'}(G)=1$. If $C_U(V)$ is non-trivial, then P must be elementary abelian since $C_U(V)=C_U(UV)$ is normal and so G can be embedded in $(G/C_U(V))\times(G/[V, U])$. This contradiction shows that $C_U(V)$ is trivial.

(5) Set $V_1=[V, W]$. Then $H=WV_1U$ is a normal subgroup of G whose index is a p'-number. Since $t(G)=t(H)$ (see [11]) and $P=WU$,

we have $G=H$ and so $V=[V, W]$.

(6) The assertion follows from $C_G(U)=U$.

(7) Let U_1 be a minimal normal subgroup of UV contained in U. We prove first that U_1 is normal in G. Assume that $U_1^\sigma \neq U_1$ for some $\sigma \in W$. Then $U_1^\sigma \cap U_1 = 1$ since $U_1^\sigma \cap U_1$ is normal in UV. Noting that V is a p'-group and U is elementary abelian, we can easily see that there exists a normal subgroup U_2 of UV such that $U=U_1 \times U_1^\sigma \times U_2$. Since $J(KG)$ contains the kernel of the natural homomorphism $KG \to KG/U$, we obtain $J(KG) \supseteq J(KW)\hat{V}+J(KU)KG$ (see [8]). We set $U_3 = U_1^\sigma \times U_2$. Then, in view of $t(G)=\alpha(p-1)+1$, we have $0=\hat{W}\hat{V}\hat{U}_1(1-\sigma)\hat{V}\hat{U}_3 = \hat{W}\hat{V}(\hat{U}_1-\hat{U}_1^\sigma)\hat{V}\hat{U}_3 = |V|\hat{W}\hat{V}(\hat{U}_1-\hat{U}_1^\sigma)\hat{U}_3 = |V|\hat{W}\hat{V}\hat{U}_1\hat{U}_3 = |V|\hat{G}$, which is impossible. Hence U_1 is normal in G. Since V is a p'-group, we have $U=U_1 \times U_2 \times \cdots \times U_t$ where every U_i is a minimal normal subgroup of UV. Then, by the preceding argument, every U_i is a normal subgroup of G. If $t \geq 2$, then P must be elementary abelian since G can be embedded in $G/U_1 \times G/U_2 \times \cdots \times G/U_t$. This contradiction shows that $t=1$ and so U is a minimal normal subgroup of G.

In the remainder of this section, we preserve the notations used in Lemma 11. The next is the main result of this section.

Theorem 12. *If V is abelian, then G can be regarded as a permutation group on $GF(p^p)$ such that $U=\{x \to x+b \mid b \in GF(p^p)\}$, W is the Galois group of $GF(p^p)$ over $GF(p)$ and that $V \subseteq \{x \to tx \mid t \in \langle \lambda^{p-1} \rangle\}$ where λ is a generator of the multiplicative group of $GF(p^p)$.*

Proof. By virture of Lemma 11 (7) and the proofs of [9, Proposition 19.8 and Theorem 19.11], we can regard G as a permutation group on some $GF(p^n)$ such that $U=\{x \to x+b \mid b \in GF(p^n)\}$, W is a subgroup of the Galois group of $GF(p^n)$ over $GF(p)$ and that V is a subgroup of $\{x \to ax \mid a \in GF(p^n)^*\}$. Since W is elementary abelian and cyclic, it follows that W is of order p. Thus $n=pr$ with some integer r, and $p^{2p+1} > p^{p+1}(p^p-1)/(p-1) \geq |G| > |W|\ |U| = p^{rp+1}$, which implies $r=1$. Now, by Lemma 11 (5), we can easily see that the order of V divides $(p^p-1)/(p-1)$. This completes the proof.

In [10], Y. Tsushima stated that if H is a p-solvable group with a regualr p-Sylow subgroup S of order p^β and $t(H)=\beta(p-1)+1$, then S is elementary abelian. On page 37, line 11 [10], he claimed that since P is of exponent p, G is of p-length 1 by Hall Higman's theorem. However, unfortunately, Tsushima's argument is unjustifiable for Fermat primes. The next shows that Tsushima's result holds under extra assumption that $O_{p',p,p'}(H)/O_{p',p}(H)$ is abelian.

Corollary 13. *Let H be a p-solvable group with a p-Sylow subgroup S of order p^β and $O_{p',p,p'}(H)/O_{p',p}(H)$ abelian. If $t(H) = \beta(p-1)+1$ and S is regular, then S is elementary abelian.*

Proof. Let H be a counter example of the minimal order. Then S is not regular by Theorem 12 and [8, Lemma 4]. Hence S is elementary abelian.

4. In this section, for $p=3$, we shall give an example of a group H which possesses all the properties listed in Lemma 11 but satisfies $t(H) > \beta(p-1)+1$ where p^β is the order of a p-Sylow subgroup of H.

We set $a = \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}$, $b = \begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix}$, and $c = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ in $M = SL(2, 3)$. Then $a^4=1$, $b^2=a^2$, $b^a(=a^{-1}ba)=b^{-1}$, $c^3=1$, $a^c=a^3b$, $b^c=a$, and $Q=\langle a, b\rangle$ is a normal 3'-subgroup of M. Let H be a semi-direct product of $U = \langle x, y \mid x^3 = y^3 = 1, xy = yx\rangle$ by M with respect to the identity map of M. Then we have the following relations:

$$x^a = y^2,\ y^a = x,\ x^b = xy^2,\ y^b = x^2y^2,\ x^c = x,\ y^c = xy.$$

It is easy to see that H possesses all the properties listed in Lemma 11. We set $a\circ b = a+b-ab$, $\chi = 1+x+x^2$, $\nu = 1+y+y^2$, $f = a^2-1$ and $\tau = c(1+a\circ b)f$ in the group algebra KH. Then we have the following

Lemma 14. (1) *f is a central idempotent in KM.*

(2) *$a^2f = -f$, $fyf = fy^2f$, and $f\chi = \chi f$.*

(3) *τ is central in KM. In paticular, c commutes with $(a\circ b)f$.*

(4) *$\tau^2 = c^2(1-a\circ b)f$ and $\tau^3 = f$.*

(5) *$J(KH) \supseteq J(KT)$ where KT is the group algebra of a cyclic group $T = \{f, \tau, \tau^2\}$ over K.*

Proof. (1)—(4) can be proved by direct verification.

(5) It follows from (3) that KT is contained in the center of KM. Since $J(KH)$ contains the kernel of the natural homomorphism $KH \to KH/U = KM$, we obtain $J(KH) \supseteq J(KM) \supseteq J(KT)$.

Lemma 15. $\chi f(y+\tau y\tau^2+\tau^2 y\tau)f = -\hat{U}f(1+b+ab)$.

Proof. By making use of Lemma 14, we obtain the following equations:

$$\begin{aligned}\chi f\tau y\tau^2 f &= \chi c(1+a\circ b)fyc^2(1-a\circ b)f\\ &= \chi(1+a\circ b)fc^{-2}yc^2(1-a\circ b)f\\ &= \chi f(1-b^{-1}\circ a^{-1})x^2yf(1-a\circ b)\\ &= \chi f(x^2y+(x^2y)^a a+(x^2y)^b b-(x^2y)^{ab}ab)f(1-a\circ b)\\ &= f\chi(x^2y+xya+xb-yab)f(1-a\circ b)\\ &= \chi f(y+ya+b-yab)(1-a-b+ab)f\\ &= \chi f(1+a+b+ab-ya-yb-yab)f.\end{aligned}$$

$$\begin{aligned}\chi f\tau^2 y\tau f &= \chi c^2(1-a\circ b)fyc(1+a\circ b)f\\ &= \chi(1-a\circ b)fy^c(1+a\circ b)f\\ &= \chi f(1+b^{-1}\circ a^{-1})xyf(1+a\circ b)\\ &= \chi f(xy-(xy)^a a-(xy)^b b+(xy)^{ab}ab)f(1+a\circ b)\\ &= f\chi(xy-xy^2a-yb+x^2ab)f(1+a\circ b)\\ &= \chi f(y-y^2a-yb+ab)f(1+a\circ b)\\ &= \chi f(y-ya-yb+ab)(1+a+b-ab)f\\ &= \chi f(1-a+b+ab+ya-yb-yab)f.\end{aligned}$$

Thus, from those above we get

$$\begin{aligned}\chi f(y+\tau y\tau^2+\tau^2 y\tau)f &= \chi f(y-1)(1+b+ab)f\\ &= -\chi f\nu f(1+b+ab) = -\widehat{U}f(1+b+ab).\end{aligned}$$

We are now ready to establish $t(H)>7=3(3-1)+1$.

Proposition 16. $t(H)\geq 9$.

Proof. It follows from Lemma 14 (5) that $J(KH)^8$ contains an element $\chi\widehat{T}\nu\widehat{T}$ where $\widehat{T}=f+\tau+\tau^2$. By Lemma 15, we have

$$\begin{aligned}\chi\widehat{T}\nu\widehat{T} &= \chi\widehat{T}(fyf+fy^2f)\widehat{T} = -\chi\widehat{T}fyf\widehat{T} = -\chi f\widehat{T}y\widehat{T}f\\ &= -\chi f(y+\tau y\tau^2+\tau^2 y\tau)\widehat{T}f = \widehat{U}f(1+b+ab)\widehat{T}\\ &= \widehat{U}\widehat{T}(1+b+ab) = \widehat{U}(1+c+c^2)(1+b+ab)f \neq 0.\end{aligned}$$

This completes the proof.

5. In this section, we shall prove that there exist no finite rings R satisfying the following conditions:

1) R is a finte ring of characteristic p.

2) R admits an automorphism σ of order p.

3) $\mathrm{Tr}(a)=0$ for every $a\in R$ where $\mathrm{Tr}(a)$ means the $\langle\sigma\rangle$-trace of a.

4) There exists a p'-subgroup T of the unit group of R such that T

is abelian, $\sigma(T) \subseteq T$, $T \cap R^{\sigma} = 1$ and $R = \{tc \mid t \in T,\ c \in R^{\sigma}\}$, where $R^{\sigma} = \{c \in R \mid \sigma(c) = c\}$.

Suppose to the contrary that there exists such a ring, and consider a permutation group $H = \langle U, V, W \rangle$ on R (acting on the left), where $U = \{u_r : x \to x+r \mid r \in R\}$, $V = \{v_t : x \to tx \mid t \in T\}$ and $W = \langle \sigma \rangle$. Since the condition $T \cap R^{\sigma} = 1$ implies that WV is a Frobenius group, we get $J(KH) = J(KW)\hat{V}KH + J(KU)KH$ (see Corollary 8 and [8, Proposition 3]). On the other hand, $R = \{tc \mid t \in T,\ c \in R^{\sigma}\}$ gives $H = VC_G(W)V$ (see Example 3). Thus $(J(KW)\hat{V}KH)^p = 0$ by Theorem 2 and hence $t(G) = \beta(p-1)+1$ where $|U| = p^{\beta-1}$. The condition 3) implies that $(\sigma^k u_a)^p = \sigma^{kp} u_{\mathrm{Tr}(a)} = 1$, and so $S = WU$ is of exponent p. Hence a p-Sylow subgroup S of H is regular. Then, since V is abelian, it follows from Corollary 13 that S is elementary abelian, which contradicts the fact that S is not abelian.

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DEPARTMENT OF MATHEMATICS
OKAYAMA UNIVERSITY

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