

## SOME REMARKS ON WEAKLY REGULAR MODULES

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Let  $A$  be a ring with 1, and  $M$  a right  $A$ -module. In [7],  $M$  is defined to be *locally projective*, if for any  $m \in M$  there exist  $m_i \in M$  and  $f_i \in \text{Hom}_A(M, A)$  such that  $m = \sum_{i=1}^k m_i f_i(m)$ . Following the first author [4], we call  $M$  a *weakly regular module* if  $M$  satisfies one of the following equivalent conditions: 1) For any  $m \in M$  there exist  $s_i \in \text{End}_A(M)$  and  $f_i \in \text{Hom}_A(M, A)$  such that  $m = \sum_{i=1}^n s_i(m) f_i(m)$ ; 2)  $M_A$  is locally projective and every  $\text{End}_A(M)$ - $A$ -submodule of  $M$  is ideal pure; 3)  $M_A$  is locally projective and  $TI = TI^2$  for each left ideal  $I$  of  $A$ , where  $T$  denotes the trace ideal of  $M_A$ .

In this paper we shall first consider the weak regularity of certain scalar extensions of weakly regular modules. We shall show that if  $M_A$  is weakly regular and  $B$  is a finite normalizing, separable extension of  $A$  such that  $B_A$  is flat, then  $M \otimes_A B$  is weakly regular. We shall also prove that if  $G$  is a locally finite group and  $M_A$  is a weakly regular module without  $|g|$ -torsion for all  $g \in G$ , then the right  $A[G]$ -module  $M \otimes_A A[G]$  is weakly regular. In the latter part of this paper, we deal with submodules, factor modules, extensions and direct products of weakly regular modules.

Throughout the paper,  $A$  will denote an associative ring with 1, and all the modules considered will be unital. Unadorned  $\otimes$  means  $\otimes_A$ , unless otherwise stated.

Noting that every locally projective module is flat (see [7]), as a combination of [4, Theorem 7] and [1, Lemmas 19.1 and 19.18], we readily obtain the following

**Lemma 1.** *Let  $M$  be a right  $A$ -module and let  $S = \text{End}_A(M)$ . Then the following are equivalent:*

- 1)  $M_A$  is weakly regular.
- 2)  $M_A$  is locally projective and  $M/N_A$  is flat for each  $S$ - $A$ -submodule  $N$  of  $M$ .
- 3)  $M_A$  is locally projective and for every  $S$ - $A$ -submodule  $N$  of  $M$  the functor  $\text{Hom}_S(M/N, \quad) : S\text{-Mod} \rightarrow A\text{-Mod}$  preserves injectives.

Let  $B$  be a ring extension of  $A$ . If the mapping  $\sum_j x_j \otimes y_j \rightarrow \sum_j x_j y_j$  from  $B \otimes B$  to  $B$  splits as an  $B$ - $B$ -homomorphism, we say  $B/A$  is a *sepa-*

*table extension.* A ring extension  $B/A$  is a *finite normalizing extension* if and only if there exist finitely many elements  $b_1, \dots, b_n$  in  $B$  such that  $B = \sum_{i=1}^n Ab_i$  and  $Ab_i = b_iA$  for all  $i$ .

**Lemma 2.** *Let  $B/A$  be a separable extension, and  $M$  a right  $B$ -module. If  $M_A$  is flat, then so is  $M_B$ .*

*Proof.* It is easy to see that  $M \otimes B_B$  is flat. By hypothesis,  $B$  is isomorphic to a  $B$ - $B$ -direct summand of  $B \otimes B$ . Hence  $M (= M \otimes_B B)$  is isomorphic to a direct summand of  $M \otimes B (= M \otimes_B B \otimes B)$ , which implies that  $M_B$  is flat.

We are now in a position to prove the first main theorem.

**Theorem 1.** *Let  $B/A$  be a finite normalizing, separable extension such that  $B_A$  is flat. If  $M_A$  is weakly regular, then so is  $M \otimes B_B$ .*

*Proof.* It is clear that  $M \otimes B_B$  is locally projective. Let  $B = \sum_{i=1}^n Ab_i$  and  $Ab_i = b_iA$  for all  $i$ . We set  $S = \text{End}_A(M)$  and  $T = \text{End}_B(M \otimes B)$ . We shall first show that  $(M \otimes B)/N_A$  is flat for each  $T$ - $B$ -submodule  $N$  of  $M \otimes B$ . Since  $M \otimes B_A$  is flat, it suffices to show that  $(M \otimes B)a \cap N \subseteq Na$  for all  $a \in A$  (see [1, Lemma 19.18]). By induction on  $k$ , we shall show that if  $(m_1 \otimes b_1 + \dots + m_k \otimes b_k)a \in N$  then  $(m_1 \otimes b_1 + \dots + m_k \otimes b_k)a \in Na$ . Suppose that  $(m_1 \otimes b_1)a \in N$ . Then  $b_1a = a'b_1$  with some  $a' \in A$ . Since  $M_A$  is weakly regular, we can write  $m_1a' = \sum_i s_i(m_1a')f_i(m_1a')$  with some  $s_i \in S$  and  $f_i \in \text{Hom}_A(M, A)$ . Then we see that  $m_1 \otimes b_1a = \sum_i s_i(m_1a')f_i(m_1a') \otimes b_1 = \sum_i (s_i \otimes 1)(m_1 \otimes b_1)ac_ia \in Na$ , where  $f_i(m_1)b_1 = b_1c_i$ ,  $c_i \in A$ . Now, assume that  $k > 1$  and our assertion is true for  $k-1$ . Choose  $s'_j \in S$  and  $c'_j \in A$  such that  $m_k \otimes b_ka = \sum_j (s'_j \otimes 1)(m_k \otimes b_ka)c'_ja$ . Setting  $y = \sum_j (s'_j \otimes 1)(m_1 \otimes b_1 + \dots + m_k \otimes b_k)ac'_j \in N$ , we get  $v = (m_1 \otimes b_1 + \dots + m_k \otimes b_k - y)a \in N$ . Since we can write  $v = m'_1 \otimes b_1 + \dots + m'_{k-1} \otimes b_{k-1}$  with some  $m'_i \in M$ , by induction hypothesis, there exists  $z \in N$  such that  $v = za$ . Hence  $(m_1 \otimes b_1 + \dots + m_k \otimes b_k)a = (y+z)a \in Na$ . This completes our induction. Thus  $(M \otimes B)/N_A$  is flat, and so Lemma 2 proves that it is a flat  $B$ -module. Therefore,  $M \otimes B_B$  is weakly regular by Lemma 1.

Obviously,  $\text{Mat}_n(A)$  is a finite normalizing, separable extension of  $A$ . For any monic polynomial  $f$  in  $A[X]$  with  $Af = fA$ , it is known that  $A[X]/A[X]f$  is separable over  $A$  if and only if the derivative  $f'$  of  $f$  is invertible in  $A[X]$  modulo  $A[X]f$  (see, e.g. [6, Theorem 1.8]). Hence we have the following

**Corollary 1.** *Let  $M_A$  be a weakly regular module.*

(1) *For any positive integer  $n$ ,  $M \otimes \text{Mat}_n(A)$  is a weakly regular  $\text{Mat}_n(A)$ -module.*

(2) *Let  $f$  be a monic polynomial in  $A[X]$  such that  $Af = fA$  and the derivative  $f'$  of  $f$  is invertible in  $A[X]$  modulo  $A[X]f$ , and let  $B = A[X]/A[X]f$ . Then  $M \otimes B_B$  is weakly regular.*

It is well known that a group ring  $B = A[G]$  of a finite group  $G$  is separable over  $A$  if and only if the order of  $G$  is invertible in  $A$  (see, e.g. [5, Corollary 1, p.128]). Hence, if  $M_A$  is weakly regular and the order of  $G$  is invertible in  $A$  then  $M \otimes B_B$  is weakly regular by Theorem 1. This result will be generalized as follows :

**Theorem 2.** *Let  $M_A$  be a weakly regular module, and  $G$  a locally finite group. If  $M$  has no  $|g|$ -torsion for all  $g \in G$ , then the  $A[G]$ -module  $M \otimes A[G]$  is weakly regular.*

In preparation for the proof of Theorem 2, we establish the following two lemmas which generalize [2, Propositions 2.1 and 2.2].

**Lemma 3.** *Let  $0 \rightarrow N \rightarrow M \rightarrow L \rightarrow 0$  be an exact sequence of right  $A$ -modules, and  $M^* = \text{Hom}_A(M, A)$ . If  $M_A$  is locally projective, then the following are equivalent :*

- 1)  $L_A$  is flat.
- 2)  $u \in NM^*(u)$  for all  $u \in N$ .

*Proof.* 1)  $\Rightarrow$  2). If  $u \in N$ , then  $u \in N \cap MM^*(u) = NM^*(u)$  by [1, Lemma 19.18].

2)  $\Rightarrow$  1). Let  $I$  be an arbitrary left ideal of  $A$ . If  $u \in N \cap MI$ , then  $M^*(u) \subseteq M^*(MI) \subseteq I$ , and so  $u \in NM^*(u) \subseteq NI$ . Hence, again by [1, Lemma 19.18],  $L$  is flat.

**Lemma 4.** *Let  $0 \rightarrow N \rightarrow M \rightarrow L \rightarrow 0$  be an exact sequence of right  $A$ -modules. If  $M$  is locally projective, then the following are equivalent :*

- 1)  $L_A$  is flat.
- 2) Given any  $u \in L$ , there exists  $\theta \in \text{Hom}_A(M, N)$  such that  $\theta(u) = u$ .
- 3) Given any  $u_1, \dots, u_n \in N$ , there exists  $\theta \in \text{Hom}_A(M, N)$  such that  $\theta(u_i) = u_i$  ( $1 \leq i \leq n$ ).

*Proof.* 1)  $\Rightarrow$  2). Let  $u \in N$ . By Lemma 3, we can represent  $u = \sum_{i=1}^p m_i f_i(u)$  with some  $m_i \in N$  and  $f_i \in M^*$ . Then  $\sum_{i=1}^p m_i f_i$  is a desired map.

2)  $\Rightarrow$  3). We proceed by induction on  $n$ ; assume that  $n > 1$  and 3) holds for  $k < n$ . Choose  $\theta_n \in \text{Hom}_A(M, N)$  such that  $\theta_n(u_n) = u_n$ , and let  $v_i = u_i - \theta_n(u_i)$  ( $1 \leq i \leq n-1$ ). Then, by induction hypothesis, there exists  $\theta' \in \text{Hom}_A(M, N)$  such that  $\theta'(v_i) = v_i$  ( $1 \leq i \leq n-1$ ). It is easy to see that  $\theta' + \theta_n - \theta' \theta_n$  has the desired property.

2)  $\Rightarrow$  1). Let  $u \in N$ . Since  $M$  is locally projective, we can write  $u = \sum_{i=1}^q c_i h_i(u)$  with some  $c_i \in M$  and  $h_i \in M^*$ . By hypothesis, there exists  $\theta \in \text{Hom}_A(M, N)$  such that  $\theta(u) = u$ . Thus we obtain  $u = \theta(u) = \sum_{i=1}^q \theta(c_i) h_i(u)$ , and hence  $L$  is flat by Lemma 3.

*Proof of Theorem 2.* Let  $x$  be an arbitrary element in  $M \otimes A[G]$ . Then we may write  $x = m_1 \otimes g_1 + \cdots + m_n \otimes g_n$  with some  $m_i \in M$  and  $g_i \in G$ . Since  $G$  is locally finite,  $g_1, \dots, g_n$  generate a finite subgroup  $H$  of  $G$ . As was seen in the proof of Theorem 1,  $(M \otimes A[H])/N_A$  is flat for each  $\text{End}_{A[H]}(M \otimes A[H])$ - $A[H]$ -submodule  $N$  of  $M \otimes A[H]$ . Since  $M_A$  is weakly regular, for any  $m \in M$  there exist  $s_i \in \text{End}_A(M)$  and  $f_i \in \text{Hom}(M, A)$  such that

$$|H|m = \sum_i s_i(|H|m) f_i(|H|m) = |H|^2 \sum_i s_i(m) f_i(m).$$

Since  $M$  has no  $|H|$ -torsion, we obtain  $m = |H| \sum_i s_i(m) f_i(m)$ . Therefore, the right multiplication by  $|H|$  is an automorphism of  $M_A$ . Hence this map induces an automorphism of  $(M \otimes A[H])$ . Now, let  $y$  be an arbitrary element in  $N$ . By Lemma 4, there exists an  $A$ -homomorphism  $\theta : M \otimes A[H] \rightarrow N$  such that  $\theta(yh) = yh$  for all  $h \in H$ . Then the map  $\hat{\theta} : M \otimes A[H] \rightarrow N$  defined by

$$\hat{\theta}(z) = |H|^{-1} \sum_{h \in H} \theta(zh) h^{-1}$$

is an  $A[H]$ -homomorphism with  $\hat{\theta}(y) = y$ . Hence  $(M \otimes A[H])/N_{A[H]}$  is flat again by Lemma 4, and therefore weakly regular by Lemma 1. Thus there exist  $s_i \in \text{End}_{A[H]}(M \otimes A[H])$  and  $f_i \in \text{Hom}_{A[H]}(M \otimes A[H], A[H])$  such that  $x = \sum_i s_i(x) f_i(x)$ . Identifying  $M \otimes A[G]$  with  $M \otimes A[H] \otimes_{A[H]} A[G]$ , we get  $x = \sum_i (s_i \otimes 1)(x) (f_i \otimes 1)(x)$  where  $s_i \otimes 1 \in \text{End}_{A[G]}(M \otimes A[G])$  and  $f_i \otimes 1 \in \text{Hom}_{A[G]}(M \otimes A[G], A[G])$ . This completes the proof.

In the rest of this paper, we consider the weak regularity of submodules, factor modules, extensions and of direct products of weakly regular modules.

**Theorem 3.** (1) *Let  $M_A$  be a weakly regular module, and  $N$  an  $A$ -submodule of  $M$ .*

(i) *If  $N$  is ideal pure, then it is weakly regular.*

(ii) If  $M/N_A$  is locally projective, then it is weakly regular.

(2) Let  $0 \rightarrow N \xrightarrow{\alpha} M \xrightarrow{\beta} L \rightarrow 0$  be an exact sequence of right  $A$ -modules. If  $M$  is locally projective and both  $N$  and  $L$  are weakly regular, then  $M$  is weakly regular.

*Proof.* (1) (i) Since  $M/N_A$  is flat by [1, Lemma 19.18], for each  $m \in N$  there exists  $\theta \in \text{Hom}_A(M, N)$  such that  $\theta(m) = m$  (Lemma 4). On the other hand, since  $M_A$  is weakly regular, there exist  $s_i \in \text{End}_A(M)$  and  $f_i \in \text{Hom}_A(M, L)$  such that  $m = \sum_i s_i(m)f_i(m)$ . Setting  $h_i = \theta s_i | N \in \text{End}_A(N)$  and  $f_i = f_i | N \in \text{Hom}_A(N, A)$ , we get  $m = \sum_i h_i(m)f_i(m)$ . This shows that  $N_A$  is weakly regular.

(ii) Let  $\nu : M \rightarrow L = M/N$  be the natural homomorphism. Take an arbitrary element  $u$  in  $L$ . Then,  $L_A$  being locally projective, there exist  $\psi \in \text{Hom}_A(L, M)$  such that  $\nu\psi(u) = u$ . Since  $M_A$  is weakly regular, there exist  $s_i \in \text{End}_A(M)$  and  $f_i \in \text{Hom}_A(M, A)$  such that  $\psi(u) = \sum_i s_i\psi(u)f_i\psi(u)$ . Then, we have  $u = \nu\psi(u) = \sum_i \nu s_i\psi(u)f_i\psi(u)$  where  $s_i\psi \in \text{Eod}_A(L)$  and  $f_i\psi \in \text{Hom}_A(L, A)$ .

(2) Let  $m$  be an arbitrary element in  $M$ : Since  $L$  is weakly regular and  $M$  is locally projective, we see that  $\beta(m) = \sum_i s_i\beta(m)f_i\beta(m)$  with some  $s_i \in \text{End}_A(L)$  and  $f_i \in \text{Hom}_A(L, A)$  and that  $\beta s'_i(m) = s_i\beta(m)$  with some  $s'_i \in \text{End}_A(M)$ . Then we have  $m' = m - \sum_i s'_i(m)f_i\beta(m) \in \text{Ker } \beta = \text{Im } \alpha$ . Since  $N' = \text{Im } \alpha$  is weakly regular, there exist  $t_j \in \text{End}_A(N')$  and  $h_j \in \text{Hom}_A(N', A)$  such that  $m' = \sum_j t_j(m')h_j(m')$ . On the other hand, by Lemma 4, there exists  $\theta \in \text{Hom}_A(M, N)$  such that  $\theta(m') = m'$ . Hence we have

$$m = \sum_i s'_i(m)f_i\beta(m) + \sum_j t_j\theta(m')h_j\theta(m') \in \text{End}_A(M)(m)\text{Hom}_A(M, A)(m)$$

where  $t_j\theta \in \text{End}_A(M)$  and  $h_j\theta \in \text{Hom}_A(M, A)$ . This shows that  $M_A$  is weakly regular.

**Corollary 2.** Let  $M_A$  be a locally projective module. If  $M = \sum_{i \in I} M_i$  with weakly regular  $A$ -submodules  $M_i$ , then  $M$  itself is weakly regular.

*Proof.* By [4, Proposition 3.2], the external direct sum  $E = \bigoplus_{i \in I} M_i$  is weakly regular. Hence, the homomorphic image  $M$  of  $E$  is weakly regular by Theorem 3 (1) (ii).

Following [7], we say  $A$  is *strongly left coherent* if and only if any direct product of (locally) projective right  $A$ -modules is locally projective. For example, every left Noetherian ring is strongly left coherent. We conclude this paper with the following.

**Theorem 4.** *Let  $A$  be a commutative strongly coherent ring. Then any direct product of weakly regular  $A$ -modules is weakly regular.*

*Proof.* Let  $\{M_\lambda \mid \lambda \in \Lambda\}$  be a set of weakly regular  $A$ -modules, and set  $M = \prod_{\lambda \in \Lambda} M_\lambda$ . Let  $m = (m_\lambda)$  be an arbitrary element in  $M$ . By [7, Theorem 4.2], there exist  $f_1, \dots, f_n \in M^*$  such that  $M^*(m) = \sum_{i=1}^n Af_i(m)$ . Since  $M_\lambda$  is weakly regular and  $M_\lambda^*(m_\lambda) \subseteq M^*(m)$ , for each  $\lambda \in \Lambda$  we can find  $s_i^\lambda \in \text{End}_A(M_\lambda)$  such that  $m = \sum_{i=1}^n s_i^\lambda(m_\lambda)f_i(m)$ . Now, setting  $s_i = (s_i^\lambda) \in \text{End}_A(M)$ , we have  $m = \sum_{i=1}^n s_i(m)f_i(m)$ . This shows that  $M_A$  is weakly regular.

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