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A COMMUTATIVITY THEOREM FOR s-UNITAL RINGS. II

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Throughout the present paper, R will represent a ring with center C. Let J denote the Jacobson radical of R, and D the commutator ideal of R. Given x, y in R, we set $[x,y] = xy - yx$ as usual. If x, y are elements of a multiplicative group, we write $(x,y) = x^{-1}y^{-1}xy$.

We consider the following properties:

$P_{[s]}$: Given elements $x_1, \cdots, x_s$ in R, there exists a positive integer n such that $[x_i^n, x_j^n] = 0 = [x_i^{n+1}, x_j^{n+1}]$ for all i, j.

$Q_{[2]}$: For each pair of elements x, y in R there exists a positive integer n such that $(xy)^n = (yx)^n$ and $(xy)^{n+1} = (yx)^{n+1}$.

Obviously, the hypotheses (i) and (ii) in [1, Theorem 2] imply $P_{[s]}$ for any s, and $Q_{[2]}$ is equivalent to the hypothesis 2) in [3, Theorem]. The present objective is to prove tactfully the next commutativity theorem which includes essentially [1, Theorem 2] as well as [3, Theorem].

Theorem 1. *Let R be an s-unital ring. Then the following are equivalent:*

1) *R is commutative.*
2) *R satisfies $P_{[5]}$.*
3) *R satisfies $Q_{[2]}$.*

In preparation for proving Theorem 1, we state the following

Lemma 1 (cf. [1, Theorem 1]). *Let n be a positive integer, and let a, b be elements of a group. If $(a^k, b^k) = (a^k, (ab)^k) = (b^k, (ab)^k) = 1$ for $k = n$, $n+1$, then $(a,b) = 1$.*

Proof.
$$\begin{aligned} ab &= (ab)^{n+1}(ab)^{-n} \\ &= (a^{-(n+1)}b^{n+1}(ab)^{n+1}b^{-(n+1)}a^{n+1})(a^n b^{-n}(ab)^n b^n a^{-n})^{-1} \\ &= (a^{-(n+1)}b^{n+1}ab^{-n}a^{n+1})^{n+1}(a^n b^{-n}ab^{n+1}a^{-n})^{-n} \\ &= (b^{n+1}a^{-n}b^{-n}a^{n+1})^{n+1}(b^{-n}a^{n+1}b^{n+1}a^{-n})^{-n} \\ &= (ba)^{n+1}(ba)^{-n} = ba. \end{aligned}$$

Corollary 1. *Let R be a ring with 1. If R satisfies $P_{[3]}$ then J is a commutative ideal containing D; in particular $J^2 \subseteq C$ and $D^3 \subseteq JDJ = 0$.*

Proof. Since the unit group of R is commutative by Lemma 1, it is easy to see that J is commutative, and $J^2 \subseteq C$. Obviously, $P_{[3]}$ is inherited by homomorphic images of R. So, to see that $D \subseteq J$, it suffices to show that every primitive ring R satisfying $P_{[3]}$ is commutative. Since $P_{[3]}$ implies that the idempotents are central, Jacobson's density theorem shows that R must be a division ring and hence is commutative by Lemma 1.

We are now ready to complete the proof of Theorem 1.

Proof of Theorem 1. It suffices to show that each of 2) and 3) implies 1). According to [2, Proposition 1], we may (and shall) assume that R has 1. In the subsequent proof, we shall use frequently the following well-known results: Let $x, y \in R$, and let s, t be positive integers.

(I) If $[x,[x,y]]=0$ then $[x^s,y]=sx^{s-1}[x,y]$.

(II) If $x^s y = 0 = (x+1)^t y$ then $y=0$.

2) ⇒ 1). Let $a \in J$, and $x, y \in R$. By hypothesis there exist positive integers n, k such that

$$[x_1^n,x_2^n]=0=[x_1^{n+1},x_2^{n+1}] \text{ for all } x_1, x_2 \in \{a+1, y, y+1, y+a, y+1+a\}$$

and

$$[y_1^k,y_2^k]=0=[y_1^{k+1},y_2^{k+1}] \text{ for all } y_1, y_2 \in \{x, x+1, y, y+1\}.$$

Since $J^2 \subseteq C$ by Corollary 1, we readily obtain $n[a,y^n]=[(a+1)^n,y^n]=0$ and $(n+1)[a,y^{n+1}]=0$. Furthermore,

$$\begin{aligned} ny^{2n}[a,y] &= nay^{2n+1}-ny^{n+1}ay^n \\ &= \textstyle\sum_{\nu=0}^{n} ny^{n-\nu}ay^{n+\nu+1}-\sum_{\nu=0}^{n} ny^{n-\nu+1}ay^{n+\nu} \\ &= n[\textstyle\sum_{\nu=0}^{n} y^{n-\nu}ay^{\nu},y^{n+1}] \\ &= n[(y+a)^{n+1},y^{n+1}]=0. \end{aligned}$$

Similarly, we have $n(y+1)^{2n}[a,y]=0$. By (II), from those above, $n[a,y]=0=n[a,y+1]$ follows, and hence $[a,y^{n+1}]=n[a,y^{n+1}]+[a,y^{n+1}]=(n+1)[a,y^{n+1}]=0$. Then

$$\begin{aligned} [a,y^{2n+1}] &= y^{n+1}ay^n-y^{2n+1}a \\ &= \textstyle\sum_{\nu=0}^{n-1} y^{2n-\nu}ay^{\nu+1}-\sum_{\nu=0}^{n-1} y^{2n-\nu+1}ay^{\nu} \\ &= y[y^n,\textstyle\sum_{\nu=0}^{n-1} y^{n-\nu-1}ay^{\nu}]y \\ &= y[y^n,(y+a)^n]y=0. \end{aligned}$$

This together with $[a,y^{n+1}]=0$ implies

$$y^{2n+1}[a,y]=y^{2n+1}[a,y]+[a,y^{2n+1}]y=[a,y^{2(n+1)}]=0.$$

Similarly, $(y+1)^{2n+1}[a,y]=0$. Thus, again by (II), $[a,y]=0$, which shows that $J \subseteq C$. Since $D \subseteq J \subseteq C$ by Corollary 1, (I) gives $k^2x^{k-1}y^{k-1}[x,y] = kx^{k-1}[x,y^k] = [x^k,y^k] = 0$. Then $k^2[x,y] = 0$ again by (II). Similarly, $(k+1)^2[x,y] = 0$. Since k^2 and $(k+1)^2$ are relatively prime, we conclude that $[x,y] = 0$.

3) $\Rightarrow$ 1). Let U be the unit group of R. If $u \in U$ and $x \in R$, then there exists a positive integer m such that $(u^{-1}\cdot xu)^m = x^m$ and $(u^{-1}\cdot xu)^{m+1} = x^{m+1}$, and so

$$x^m[x,u] = u\{u^{-1}x^{m+1}u-(u^{-1}x^m u)x\} = u\{(u^{-1}xu)^{m+1}-(u^{-1}xu)^m x\}$$
$$= u(x^{m+1}-x^m x) = 0;$$

similarly $(x+1)^{m'}[x,u] = 0$ with some m'. Thus, by (II), $[x,u] = 0$, namely $U \subseteq C$. Now, it is easy to see that $J \subseteq C$ and every idempotent of R is central. Since $Q_{[2]}$ is inherited by any homomorphic image of R, the argument used in the proof of Corollary 1 enables us to see that R/J is commutative.

Now, let x, y be arbitrary element of R. We claim that if $(xy)^m = (yx)^m$ and $(xy)^{m+1} = (yx)^{m+1}$ then $(xy)^k = (yx)^k$ for all $k \geq m$. In fact,

$$(xy)^m xy = (xy)^{m+1} = (yx)^{m+1} = (yx)^m yx = (xy)^m yx,$$

and so $(xy)^k xy = (xy)^k yx$ for all $k \geq m$. Thus, $(xy)^k = (yx)^k$ yields $(xy)^{k+1} = (yx)^{k+1}$. In view of this claim, we can find a positive integer n such that

$$(x_1x_2)^k = (x_2x_1)^k \text{ for all } k \geq n \text{ and } x_1,\ x_2 \in \{x,\ x+1,\ y,\ y+1\}.$$

Noting that $y^nx^n-(yx)^n \in J \subseteq C$, we get

$$y^n[x^n,y] = [y^nx^n,y] = [(yx)^n,y] = y(xy)^n - y(yx)^n = 0,$$

and similarly $(y+1)^n[x^n,y] = 0$. Hence, $[x^n,y] = 0$ by (II), and similarly $[x^{n+1},y] = 0$. From those above, we obtain $x^n[x,y] = [x^{n+1},y]-[x^n,y]x = 0$, and similarly $(x+1)^n[x,y] = 0$. We conclude therefore, again by (II), that $[x,y] = 0$.

The results presented invite the conjecture that every s-unital ring satisfying $P_{[3]}$ must be commutative.

REFERENCES

[1] H. ABU-KHUZAM and A. YAQUB: Rings and groups with commuting powers, Intern. J. Math. & Math. Sci. 4 (1981), 101–107.

[2] Y. HIRANO, Y. KOBAYASHI and H. TOMINAGA: Some polynomial identities and commutativity of s-unital rings, Math. J. Okayama Univ. 24 (1982), 7—13.
[3] M. HONGAN and H. TOMINAGA: A commutativity theorem for s-unital rings, Math. J. Okayama Univ. 21 (1979), 11—14.

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