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ON RINGS SATISFYING THE IDENTITY $(x+x^2+\cdots+x^n)^{(n)}=0$

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Throughout the present paper, R will represent a ring with center C. Let N be the set of nilpotents in R, N^* the subset of N consisting of all a with $a^2=0$, E the set of idempotents in R, and D the commutator ideal of R. For $x\in R$, we define inductively $x^{(1)}=x$, $x^{(k)}=x^{(k-1)}\circ x$, where $x\circ y=x+y+xy$. We may write formally $x^{(k)}=(1+x)^k-1$.

Let n be a positive integer, and consider the following properties:

(i)$_n$ $(x+x^2+\cdots+x^n)^{(n)}=0$ for all $x\in R$.

(ii) N is commutative.

(ii)* N^* is commutative.

(iii) $[[a,x],x]=0$ for all $a\in N$ and $x\in R$.

(iv) If $a\in N$, $x\in R$ and $[a,x]^2=0$ then $[a,x]\in C$.

(*) For any x, $y\in R$, $(x+xy)\circ(y+yx)=0$ if and only if $x=y$.

If R has 1, then (i)$_n$ becomes

(i)$'_n$ $(1+x+x^2+\cdots+x^n)^n=1$ for all $x\in R$.

The present objective is to prove the following theorems.

Theorem 1. *Let R be a left s-unital ring satisfying* (i)$_n$. *If R is normal (i.e., E is central) then N is a nil ideal and R/N is commutative.*

Theorem 2. *Let R be a left s-unital ring satisfying* (i)$_n$.

(1) *If R satisfies* (ii)*, *then N is a nil ideal and R/N is a commutative regular ring.*

(2) *If R satisfies* (ii), *then N is a commutative nil ideal and R/N is a commutative regular ring.*

(3) *If R satisfies* (ii) *and* (iii) (*or* (iv)), *then R is commutative and R/N is a regular ring.*

Theorem 3. *Let R be a left s-unital ring satisfying* (i)$_{2^k}$. *Then N is a nil ideal and $R=R_1\oplus R_2$, where R_1 is either* 0 *or a commutative regular ring of odd characteristic, $R_2\supseteq N$ and R_2/N is a Boolean ring.*

Theorem 4. *If R is a normal, left s-unital ring satisfying* (i)$_{2^k}$ *and* (ii), *then* R *is commutative and* $R=R_1\oplus R_2$, *where* R_1 *is either* 0 *or a regular ring of odd characteristic,* $R_2\supseteq N$ *and* R_2/N *is a Boolean ring.*

Theorem 5 (cf. [4, Theorem 2]). *A left s-unital ring* R *satisfies* $(*)$ *if and only if* (a) R *is commutative and* R/N *is a Boolean ring and* (b) $a^{(2)}=0$ *for all* $a\in N$.

We start with the following lemma.

Lemma 1. *Suppose that* R *satisfies* $(\mathrm{i})_n$ *and* $p^\alpha R=0$, *where* p *is a prime. Then there exists a positive integer* m *such that* $x^m=x^{2m}$ *for all* $x\in R$.

Proof. Let $y=x+x^2+\cdots+x^n$, and $n=p^\beta t$, where $\beta\geq 0$ and $(p,t)=1$. Then there exists $u(\lambda)\in \boldsymbol{Z}[\lambda]$ such that $y^{(t)}=(1+y)^t-1=tx+x^2u(x)$. Noting here that $(pR)^\alpha=0$, we can easily see that $(tx+x^2u(x))^{p^\beta\alpha}=0$. Because $(t,p)=1$, we readily obtain $x^{p^\beta\alpha}-x^{p^\beta\alpha+1}f(x)=0$ with some $f(\lambda)\in \boldsymbol{Z}[\lambda]$. Now, by making use of the argument employed in the proof of [1, Lemma], we can find a positive integer m such that $x^m=x^{2m}$ for all $x\in R$.

Remark 1. (1) Recently, Komatsu [5] proved that a ring R with 1 satisfies a polynomial identity $x^{2m}-x^m=0$ for some positive integer m if and only if the addition of R is equationally definable in terms of the multiplication and the successor operation.

(2) Let p be a prime. If $(1+n)^n\equiv 1 \pmod p$ and $p-1\mid n$, then $\mathrm{GF}(p)$ satisfies $(\mathrm{i})'_n$. For instance, $\mathrm{GF}(3)$ satisfies $(\mathrm{i})'_4$.

(3) Let R be a left s-unital ring satisfying $(\mathrm{i})_n$. Let x be an arbitrary non-zero element of R, and choose $e\in R$ such that $ex=x$. Then, by $(\mathrm{i})_n$,

$$0=\{(1+e+e^2+\cdots+e^n)^n-1\}x=\{(n+1)^n-1\}x.$$

This means that the characteristic of R is non-zero. Furthermore, n has to be even. In fact, if n is odd, then

$$0=\{(1-e+e^2-\cdots-e^n)^n-1\}x=-x,$$

which is a contradiction. In particular, if p is a prime and $n=p^k$ then $p=2$.

Proof of Theorem 1. By Remark 1 (3), the characteristic of R is non-zero. Obviously, the hypothesis $(\mathrm{i})_n$ and the normality are inherited by subrings, so we may assume that the additive group of R is a p-group, and therefore $p^a R = 0$ for some prime p. Then, by Lemma 1, there exists a positive integer m such that $x^m = x^{2m}$ for all $x \in R$, and so R satisfies the polynomial identity $[x^m, y] = 0$. Since $[E_{11}, E_{11} + E_{12}] = E_{12} \neq 0$ in $(\mathrm{GF}(q))_2$ (q a prime), D is a nil ideal by [3, Proposition 2].

Remark 2. For $e \in E$, the following are equivalent:

1) $e \in C$.

2) $[e, a] = 0$ for all $a \in N^*$.

3) $[e, [e, a]] = 0$ for all $a \in N^*$.

4) $[(e+a)e, e(e+a)] = 0$ for all $a \in N^*$.

5) $[e, f] = 0$ for all $f \in E$.

In fact, it is clear that 1) implies 2)—5), and 2) does 3). In order to see that each of 3)—5) implies 1), given $x \in R$, we set $a = ex(1-e) \in N^*$. It is easy to see that if 3) or 4) is satisfied then $a = 0$, i.e., $ex = exe$. Similarly, we can see that $xe = exe$. Finally, if 5) is satisfied, then $e + a \in E$ and $e + a = e(e+a) = (e+a)e = e$, whence it follows again $a = 0$, i.e., $ex = exe$; similarly $xe = exe$. In particular, R is normal if and only if E is commutative, and (iii) implies the normality of R. Furthermore, if N^* is central then R is normal and N coincides with the prime radical of R. In fact, if $a^n = 0$ and $a^{n-k}(Ra)^{k+1} = 0$ then $\{a^{n-k-1}(Ra)^{k+1}\}^2 \subseteq a^{n-k-1}Ra^{n-k}(Ra)^{k+1} = 0$. Hence $a^{n-k-1}(Ra)^{k+1} \subseteq C$, and so $a^{n-k-1}(Ra)^{k+2} = Ra^{n-k}(Ra)^{k+1} = 0$. We get eventually $(Ra)^{n+1} = 0$.

In advance of proving Theorem 2, we state the next lemma.

Lemma 2. (1) *If R satisfies* (ii)* *then R/P is normal, where P is the prime radical of R.*

(2) *If a π-regular ring R satisfies* (ii)*, *then N coincides with the Jacobson radical of R and R/N is strongly regular.*

Proof. (1) Let $\bar{e}$ be an arbitrary idempotent of $\bar{R} = R/P$. Since P is a nil ideal, we may assume from the beginning that e is in E. By hypothesis, $eR(1-e)\cdot(1-e)Re = (1-e)Re\cdot eR(1-e)$, and so $eR(1-e)Re = 0$. Hence $\bar{e}\bar{R}(1-\bar{e})\bar{R}\bar{e} = 0$. By the semiprimeness of $\bar{R}$, we get $\bar{e}\bar{R}(1-\bar{e}) = 0$, and therefore $\bar{e}\bar{x} = \bar{e}\bar{x}\bar{e}$ for all $x \in R$. Furthermore, $\bar{e}\bar{R}(1-\bar{e})\bar{R} = 0$ yields $(1-\bar{e})\bar{R}\bar{e} = 0$, and therefore $\bar{x}\bar{e} = \bar{e}\bar{x}\bar{e}$ for all $x \in R$. Thus, we have seen that $\bar{e}$ is central.

(2) Let J be the Jacobson radical of the π-regular ring R. Obviously, J/P is a nil ideal of R/P. Since $R/J \simeq (R/P)/(J/P)$ and R/P is normal by (1), it is easy to see that R/J is normal and any nilpotent element of

R/J generates a nil right ideal. Hence, R/J is reduced and N coincides with J. The reduced π-regular ring R/N is strongly regular (see, e.g. [2]).

Proof of Theorem 2. (1) As in the proof of Theorem 1, we may assume that $p^a R = 0$ with some prime p. Then, by Lemma 1, there exists a positive integer m such that $x^m = x^{2m}$ for all $x \in R$. Hence, by Lemma 2 (2), N coincides with the Jacobson radical of R and R/N satisfies the identity $x - x^{m+1} = 0$. By Jacobson's commutativity theorem, R/N is commutative.

(2) This is clear by (1).

(3) By the proof of (1) and [6, Theorem 1], R is commutative.

Lemma 3. *If R satisfies* $(\mathrm{i})_{2^k}$ *and* $2^a R = 0$, *then N is a nil ideal and R/N is a Boolean ring.*

Proof. Let $n = 2^k$. Obviously, $y = x + x^2 + \cdots + x^n \in N$, and so $x - x^{n+1} = y - xy \in N$. Then, noting that $x(1-x)^n - (x - x^{n+1}) \in 2R \subseteq N$, we readily see that $x - x^2 = x(1-x) \in N$. Now, we claim that for any prime q, $(\mathrm{GF}(q))_2$ fails to satisfy $(\mathrm{i})'_n$. If $q \neq 2$ then $x = E_{12}$ does not satisfy $(\mathrm{i})'_n$. On the other hand, if $(\mathrm{GF}(2))_2$ satisfies $(\mathrm{i})'_n$ then, as we have seen just above, $x - x^2$ is nilpotent for all $x \in (\mathrm{GF}(2))_2$. But this is not true for $x = E_{11} + E_{12} + E_{21}$. Thus, by [3, Proposition 2], D is a nil ideal, and so R/N is a Boolean ring.

Proof of Theorem 3. By Remark 1 (3), $hR = 0$ for some positive integer $h = 2^a h'$, $(2, h') = 1$. Then, $R = R_1 \oplus R_2$, where $h' R_1 = 0$ and $2^a R_2 = 0$. If a is an element of R_1 with $a^2 = 0$ then, by $(\mathrm{i})_{2^k}$, we can easily see that $2^k a = 0$, and therefore $a = 0$. This proves that R_1 is a reduced ring and $N \subseteq R_2$. Now, the assertion is an easy combination of Theorem 2 (1) and Lemma 3.

Proof of Theorem 4. In view of Theorem 3, it remains only to prove that R_2 is commutative. Obviously, every element of R_2 is of the form $e + a$, where $e \in E \cap R_2$ and $a \in N$. Since E is central and N is commutative, it is immediate that R_2 is commutative.

Remark 3. As the following example shows, Theorem 4 is not true if we replace 2^k by an arbitrary positive integer: Let

$$R = \left\{ \begin{pmatrix} a & b & c \\ 0 & a^2 & 0 \\ 0 & 0 & a \end{pmatrix} \mid a,\ b,\ c \in \mathrm{GF}(4) \right\}.$$

Then $12R=0$ and R is a normal ring satisfying (i)$'_{12}$ and (ii). But R is not commutative and R/N is not Boolean either.

Proof of Theorem 5. "Only if": Obviously, (*) implies (i)$_2$. We claim that R is normal. To see this, given $e \in E$ and $x \in R$, we put $a=ex(1-e)$. Since

$$\{(a-e)-(a-e)e\}\circ\{-e-e(a-e)\}=a\circ(-a)=-a^2=0,$$

(*) shows that $a-e=-e$, and so $a=0$, i.e., $ex=exe$. A similar argument gives $xe=exe$, and hence $ex=xe$. It is easy to see that $8R=0$, and therefore R is a normal, left s-unital ring with $2^3R=0$ and satisfies (i)$_2$. We shall show that R satisfies (ii). Let x be an arbitrary element of R, and let e be such that $ex=x$. Since $(x+x^2)^{(2)}=0=(-x+x^2)^{(2)}$ by (i)$_2$, we readily obtain $4(x+x^3)=0$, i.e., $4x=4x^3$. Replacing x by $e+x$ in the last, we get

$$4x+4x^2=4(e+x)x=4(e+x)^3x=4x+4x^2+4x^3-4x^4.$$

Combining this with $4x=4x^3$, we see that $4x=4x^4=4x^3x=4x^2$. Since $x^4=-2x^3-3x^2-2x$ by (i)$_2$, we get

$$x^5=-2x^4-3x^3-2x^2=-2(-2x^3-3x^2-2x)-3x^3-2x^2=x^3+4x^2-4x=x^3.$$

This implies that $a^3=0$ for all $a \in N$, and therefore

$$a+a^2=(a+a^2)^{(2)}\circ(-a)=-a, \text{ i.e., } a^{(2)}=0.$$

Noting that N is a nil ideal by Theorem 1, we get for any $a,\ b \in N$

$$a\circ b=a\circ(a\circ b)^{(2)}\circ b=b\circ a,$$

which shows that N is commutative. Thus, by Theorem 4, R is commutative and R/N is a Boolean ring.

"If": First, we claim that every quasi-regular element of R is nilpotent. In fact, if a is quasi-regular then the nilpotency of $a+a^2$ yields that of a. Obviously, $(x+x^2)^{(2)}=0$ for all $x \in R$. Conversely, if $(x+xy)\circ(y+yx)=0$ then $x+xy$ is nilpotent, and hence $y+xy=(x+xy)^{(2)}\circ(y+yx)=x+xy$, whence it follows that $y=x$.

Remark 4. In order to prove the only if part of Theorem 5, we quoted Theorems 1 and 4. In case R has 1, we can prove it more directly. In fact, $x^5=x^3$ yields that $u^2=1$ for every unit u in R. If u, v are units in R then $uv=(uv)^{-1}=v^{-1}u^{-1}=vu$. In particular, if a and b are in N then $[a,b]=[1+a,1+b]=0$, and so N is commutative. Now, let $a \in N$, $x \in R$. Since $(x+x^2)^6=-8(x+x^2)^3=0$ and $(x^2+x^4)^6=0$ by (i)$'_2$ and x^4

is a central idempotent, we have $[a,x] = [a,x-x^4] = [a,x+x^2]-[a,x^2+x^4] = 0$, and hence $N \subseteq C$. Therefore, $x = (x+x^2)-(x^2+x^4)+x^4 \in C$ for all x in R, and hence R is commutative.

REFERENCES

[1] H. ABU-KHUZAM, H. TOMINAGA and A. YAQUB: Equational definability of addition in rings satisfying polynomial identities, Math. J. Okayama Univ. **22** (1980), 55—57.

[2] Y. HIRANO: Some studies on strongly π-regular rings, Math. J. Okayama Univ. **20** (1978), 141—149.

[3] Y. HIRANO, Y. KOBAYASHI and H. TOMINAGA: Some polynomial identities and commutativity of s-unital rings, Math. J. Okayama Univ. **24** (1982), 7—13.

[4] Y. HIRANO, H. TOMINAGA and A. YAQUB: Some polynomial identities and commutativity of s-unital rings. II, Math. J. Okayama Univ. **24** (1982), 111—115.

[5] H. KOMATSU: On the equational definability of addition in rings, Math. J. Okayama Univ. **24** (1982), 133—136.

[6] H. TOMINAGA and A. YAQUB: Some commutativity properties for rings, Math. J. Okayama Univ. **25** (1983), 81—86.

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