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CONDITIONS FOR ELEMENTS TO BE CENTRAL IN CERTAIN RINGS

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Recently, many papers, [3], [4], [7], [9], [10] and others, have been devoted to studying the conditions which force an element of a ring to be central. In this paper, we continue the investigation on such conditions, and partially generalize the results of Awtar [2], Mogami and Hongan [10] and Felzenszwalb [3].

Throughout this paper R is an associative ring with 1, and Z the center of R. Given a subset M of R, $r(M)$ and $l(M)$ denote the right annihilator and the left annihilator of M in R, respectively.

Now, let a and x be elements of R, and consider the following conditions:

(i) $[ax,xa] \in Z$.

(ii) There exists an integer $n(x)>1$ such that

$$(xa)^k - x^k a^k \in Z, \quad k=n(x),\ n(x)+1,\ n(x)+2.$$

The main theorems of this paper are stated as follows:

Theorem 1. *Let R be a semiprime ring, and U an ideal of R with $l(U)=0$. If (i) is satisfied for all $x \in U$, then a is central.*

Theorem 2. *Let R be a prime ring with no non-zero nil left ideals and $2R \neq 0$. If (ii) is satisfied for all $x \in R$, then a is central.*

In preparation for proving our theorems, we establish the following lemmas.

Lemma 1. *Let R be a division ring. If (i) or (ii) is satisfied for all $x \in R$, then a is central.*

Proof. Suppose, to the contrary, $a \notin Z$. If (i) is satisfied then $[x,a^{-1}xa] \in Z$ for all $x \in R$. Next, assume that (ii) is satisfied. Since

$$x[x^{k-1}a^{k-1},ax]a = [x^k a^k,xa] = [x^k a^k-(xa)^k,xa] = 0,$$

we get $[x^{k-1}a^{k-1},ax]=0$, and hence

$$[(xa)^{k-1},ax] = [(xa)^{k-1}-x^{k-1}a^{k-1},ax] = 0, \quad k=n(x)+1,\ n(x)+2.$$

From the last, we obtain $(xa)^n[xa,ax] = [(xa)^{n+1},ax]-[(xa)^n,ax]xa = 0.$

This proves that $[xa,ax]=0$, and so $[a^{-1}xa,x]=0$. Thus, in any case, $[x,a^{-1}xa]\in Z$. Since the inner automorphism effected by a is non-trivial, R is commutative by [11, Remark 2]. This contradiction shows that $a\in Z$.

Lemma 2. *Let R be a primitive ring, and U a non-zero ideal of R. If* (i) *or* (ii) *is satisfied for all $x\in U$, then a is central.*

Proof. Since R is primitive, it has a faithful irreducible module V which can also be regarded as a faithful irreducible U-module. By Density Theorem, U acts densely on V as a vector space over a division ring Δ. If $\dim_\Delta V=1$ then R is a division ring, and $a\in Z$ by Lemma 1. So, we assume henceforth that $\dim_\Delta V>1$. Let v be a non-zero vector in V. We claim that v and va are linearly dependent. First, consider the case of (i). Suppose that v, va, va^2 are linearly independent. By Density Theorem, there exist u_1, $u_2\in U$ such that $vu_1=v$, $(va)u_1=v$, $(va^2)u_1=0$; $vu_2=0$, $(va)u_2=-v$, $(va^2)u_2=va$. Thus, $0=v[[u_1a,au_1],u_2]=v$, which contradicts the choice of v. So we may assume that $va^2=\beta v+\gamma va$, where β, $\gamma\in\Delta$. If v and va are linearly independent, then there are u_1', $u_2'\in U$ such that $vu_1'=v$, $(va)u_1'=v$; $vu_2'=0$, $(va)u_2'=-v$, which yields again $v=0$. This proves that v and va are linearly dependent. Next, assume that (ii) is satisfied for all $x\in U$. If v and va are linearly independent then there are w_1, $w_2\in U$ such that $vw_1=0$, $(va)w_1=v$; $vw_2=va$, $(va)w_2=v$. Thus,

$$0=v[(w_1a)^k-w_1^ka^k,w_2]=-va \quad \text{for } k=n(w_1).$$

This contradiction shows that v and va are linearly dependent. Thus, in any case, for every $v\in V$ we have $va=\lambda(v)v$, where $\lambda(v)\in\Delta$. Noting that $\dim_\Delta V>1$, we can easily see that $\lambda(v)$ does not depend on v, i.e., $va=\lambda v$ for all $v\in V$. Now, if $u\in U$, we have $(vu)a=\lambda vu=(va)u$. Thus $V[u,a]=0$, that is $[u,a]=0$ for all $u\in U$. Now, by [8, Lemma 1.1.6], we get $a\in Z$.

Lemma 3. *Let R be a semiprimitive ring, and U a two-sided ideal of R with $l(U)=0$. If* (i) *or* (ii) *is satisfied for all $x\in U$, then a is central.*

Proof. Divide the set of all primitive ideals of R into two parts: let $\mathcal{P}_1$ be the set of those which contain U, and $\mathcal{P}_2$ the set of those which do not. Let $U_1\equiv\bigcap_{P\in\mathcal{P}_1}P$, and $U_2\equiv\bigcap_{P\in\mathcal{P}_2}P$. Then $U_1\cap U_2=0$. Since $U_2\subseteq l(U_1)\subseteq l(U)=0$, R is a subdirect sum of R/P $(P\in\mathcal{P}_2)$. Obviously, $(U+P)/P$ is non-zero for every $P\in\mathcal{P}_2$. Hence, by Lemma 2, $[a,x]\in\bigcap_{P\in\mathcal{P}_2}P=0$ for all $x\in R$.

Corollary 1. *Let R be a semiprimitive ring. If* (i) *or* (ii) *is satisfied for all $x \in R$ then a is central.*

Lemma 4. *Let R be a ring without non-zero nil left ideals, and $a \in R$. Suppose that for each $x \in R$ there exists $n = n(x) > 1$ such that $(xa)^n - x^n a^n \in Z$.*

(1) *Let $b \in l(a)$. If $x, y \in R$ and $xy = 0$ then $xa^k by = 0$ for all $k > 1$.*

(2) $l(a) = \{x \in R \mid xa^m = 0$ *for some* $m \geq 1\}$.

(3) $r(a) = l(a)$.

Proof. (1) For any $r \in R$, there exists $n > 1$ such that $(yrxa)^n = (yrxa)^n - (yrx)^n a^n \in Z$, and therefore $(yrxa)^{n+1} = [(yrxa)^n, yrx]a = 0$. Thus, $Rxay$ is a nil left ideal, and so we get $xay = 0$. Since $(ab)^2 = 0$, $t = 1 + ab$ is invertible and $tat^{-1} = a - a^2 b$. By the above, $xay = 0 = t^{-1}xt \cdot a \cdot t^{-1}yt$, and so we get

$$xa^2 by = -x(a - a^2 b)y = -t(t^{-1}xt \cdot a \cdot t^{-1}yt)t^{-1} = 0.$$

Now, it is easy to see that $xa^k by = 0$ for all $k > 1$.

(2) It suffices to show that $xa^m = 0$ $(m > 1)$ yields $xa^{m-1} = 0$. Putting $y = xa^{m-2}$, we have $ya^2 = 0$. For any $r \in R$, there exists $n > 1$ such that $(rya)^n = (rya)^n - (ry)^n a^n \in Z$, and therefore $(rya)^{n+1} = ry[a, (rya)^n] = 0$. Since R has no non-zero nil left ideals, we get $xa^{m-1} = ya = 0$.

(3) Let $x \in r(a)$, and $n = n(x)$. Since $x^n a^n = x^n a^n - (xa)^n \in Z$, there holds $x^{n+1} a^n = [x, x^n a^n] = 0$, whence it follows $x^{n+1} a = 0$ by (2). We consider the right ideal V consisting of all $v \in R$ such that $r^{n(r)} v = 0$ for all $r \in r(a)$. If $r \in r(a)$ and $r^2 = 0$ then, for any $v \in V$, $(rv)^m v = 0$ and $(r + rv)^m v = 0$, where $m = \max\{n(rv), n(r + rv)\}$. Expanding the last equation, we get $(rv)^m = 0$. Then, since R contains no nil right ideals either, it follows that $rV = 0$. Let y be an arbitrary element of V. Since $(x^n yx)^2 = 0$ and $x^n yx \in r(a)$, by what was just proved above, there holds $x^n yxy = 0$. Repeating the above argument, we obtain $x^{n-1} yxyxyxy = 0$, and eventually $(xy)^{1+2+\cdots+2^{n-1}+1} = 0$. Thus, xV is a nil right ideal, and hence zero; in particular, $xa = 0$. This proves $r(a) \subseteq l(a)$.

To prove the converse inclusion, let $b \in l(a)$. Let x be an arbitrary element of R, and $n = n(x)$. Then

$$0 = [x^n a^n - (xa)^n, xa] = [x^n a^n, xa] = x(x^{n-1} a^n xa - ax^n a^n).$$

By (1),

$$0 = xa^{n+2} b(x^{n-1} a^n xa - ax^n a^n) = xa^{n+2} bx^{n-1} a^n xa,$$

and therefore again by (1)

$$(bxa^{n+2})^{n+1} = bxa^{n+2}b(xa^{n+2}b)^{n-2}xa^n \cdot a^2b \cdot xa \cdot a^{n+1} = 0.$$

Recalling that R contains no non-zero nil left ideals, we obtain $l(a)xa^{n+2} = 0$, and therefore $l(a)xa = 0$ by (2). Then, since $(xab)^2 = 0$, we see that Rab is a nil left ideal, and hence $ab = 0$, which proves $l(a) \subseteq r(a)$.

We are now ready to complete the proofs of our theorems.

Proof of Theorem 1. As in the proof of Lemma 3, we see that there exist prime ideals P_α of R such that $\bigcap_\alpha P_\alpha = 0$ and $U \nsubseteq P_\alpha$. Thus, without loss of generality, we may assume that R is a prime ring and U is a non-zero ideal of R. Let $S = RC$ be the central closure of R, where C is the extended centroid of R (see [8, pp. 20—31]). Let $V = UC$. Noting that for any $u_1, u_2 \in U$,

$$[ax_1,x_2a]+[ax_2,x_1a] = [a(x_1+x_2),(x_1+x_2)a]-[ax_1,x_1a]-[ax_2,x_2a] \in Z,$$

we can easily see that the condition (i) carries over to S with respect to the ideal V. Suppose now that a is not in Z, and choose $u \in U$ such that $ua \neq 0$. Then, S satisfies the generalized polynomial identity $[[x_1ua,ax_1u],x_2] = 0$, and a theorem of Martindale [8, Theorem 1.3.2] shows that S is primitive, and therefore $a \in Z(S)$ by Lemma 2. Hence, $a \in Z$, a contradiction.

Corollary 2. (1) *Let R be a ring without non-zero nil ideals. If for each $x \in R$ there exists an integer $m = m(x) > 1$ such that $[x^my,yx^m] \in Z$ for all $y \in R$, then R is commutative.*

(2) *Let R be a semiprime ring. If for each $x \in R$ there exists a polynomial $p(t) = p_x(t)$ with integer coefficients such that*

$$[(x-x^2p(x))y,y(x-x^2p(x))] \in Z \quad \textit{for all } y \in R,$$

then R is commutative.

Proof. (1) Since R is a subdirect sum of prime rings without non-zero nil ideals, it is enough to prove the assertion for prime rings without non-zero nil ideals. Then, by Theorem 1, $x^m \in Z$. Hence, by [5, Theorem 5], R is commutative.

(2) By Theorem 1 and [6, Theorem 19].

Proof of Theorem 2. We may assume that a is non-zero. Furthermore, by Corollary 1, we may assume that the Jacobson radical J of R is non-zero. Let x be an arbitrary element of J. Then both $1+x$ and $1-x$ are units in R. Since R is prime and $r(a) = l(a)$ by Lemma 4 (3), a is regular. Thus, we can use the same argument as in the proof of Lemma 1 to show that

$$[a,ax]+[xa,a]+[xa,ax]=[(1+x)a,a(1+x)]=0, \text{ and}$$
$$-[a,ax]-[xa,a]+[xa,ax]=0.$$

Since $2R \neq 0$, the last two equations give $[xa,ax]=0$. Thus from Theorem 1 it follows that $a \in Z$.

As an easy consequence of Corollary 1 and Theorem 2, we have

Corollary 3. *Let R be either a semiprimitive ring or a 2-torsion free prime ring with no non-zero nil left ideals.*

(1) *If for every x, $y \in R$ there exist integers $m=m(x)>1$ and $n=n(x,y)>1$ such that*

$$(yx^m)^k - y^k x^{mk} \in Z, \quad k=n,\ n+1,\ n+2,$$

then R is commutative.

(2) *If for every x, $y \in R$ there exists a polynomial $p(t)=p_x(t)$ with integer coefficients and an integer $n=n(x,y)>1$ such that*

$$(y(x-x^2p(x)))^k - y^k(x-x^2p(x))^k \in Z, \quad k=n,\ n+1,\ n+2,$$

then R is commutative.

We conclude this paper with the following remark.

Remark. Careful scrutiny of the proof of Lemma 4 shows that, in case $n(x)$ are bounded, the lemma turns out to be true if we just assume that R has no non-zero nil left ideals of bounded index. By a result of Levitzki, every semiprime ring has no non-zero nil left ideals of bounded index. Hence, in case (ii) is satisfied for all x and $n(x)$ are bounded, Theorem 2 is still true without the hypothesis that R has no non-zero nil left ideals.

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