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THE S^1-TRANSFER MAP AND HOMOTOPY GROUPS OF SUSPENDED COMPLEX PROJECTIVE SPACES

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1. Introduction. We denote by P^n the complex n-dimensional projective space. Let $SU(n)$ and $SO(n)$ be the special unitary and rotation groups, respectively. As is well known, the suspended space EP^{n-1} is canonically embedded in $SU(n)$ [17]. We consider the composition of the canonical mappings

$$EP^{n-1} \longrightarrow SU(n) \longrightarrow SO(2n) \longrightarrow \Omega^{2n}S^{2n},$$

where $\Omega^{2n}S^{2n}$ denotes the space of base point preserving maps from S^{2n} to S^{2n}. We define the S^1*-transfer map*

$$g_n : E^{2n+1}P^{n-1} \longrightarrow S^{2n}$$

by taking the adjoint of the above.

We denote by $B_{k,n}$ the image of the induced homomorphism

$$g_{n*} : \pi_{2n+k}(E^{2n+1}P^{n-1}) \longrightarrow \pi_{2n+k}(S^{2n})$$

and

$$B_k = \lim_{n\to\infty} B_{k,n}.$$

The purpose of the present paper is to determine the group structure of B_k for $k \leq 11$.

Our method is essentially to compute stable homotopy groups of EP^{n-1} from the unstable viewpoint. The main tools are the works on the homotopy groups of $SU(n)$ [13] and the composition methods in the homotopy groups of S^n [15].

Our main result is stated as follows:

Theorem 1.1. *B_k for $k \leq 11$ and generators of their 2-primary components are listed in the following table.*

$k=$	1, 2	3	4, 5	6	7	8	9	10	11
$B_k \approx$	0	Z_{24}	0	Z_2	Z_{120}	Z_2	Z_2+Z_2	Z_3	Z_{504}
gen. of 2-comp.		ν		ν^2	2σ	$\bar{\nu}$	$\eta^2\sigma,\ \nu^3$		ζ

Let $\pi_n : S^{2n+1} \longrightarrow P^n$ be the natural projection. The following theorem plays an important role in proving Theorem 1.1.

Theorem 1.2. *The homotopy class of* $E^{k+1}\pi_{n-1}$ *is of order* $n!$ *for* $k \geq 0$.

The result about the k-th stable homotopy group of EP^∞ for $k \leq 11$ will be stated as Theorem 1.3 in the last section.

2. The S^1-transfer map and the complex J homomorphism. Let S^{2n-1} be the unit sphere in the complex n-dimensional space C^n and let $U(n)$ be the unitary group. Let $\phi: S^{2n-1}\times S^1 \longrightarrow U(n)$ be a mapping defined as follows: $\phi(u,q)v=v+u(q-1)\langle u,v\rangle$ for $u \in S^{2n-1}$, $q \in S^1$ and $v \in C^n$, where $\langle u,v\rangle=\sum_{k=1}^n \bar{u}_k v_k$ for $u=(u_1,\cdots,u_n)$ and $v=(v_1,\cdots,v_n)$. Then it gives the matrix form $\phi(u,q)=(\delta_{ij}+(q-1)\bar{u}_i u_j)$ for $1 \leq i, j \leq n$.

According to [5], the complex quasi-projective space EP_+^{n-1} is the space obtained from $S^{2n-1}\times S^1$ by imposing the equivalence relation: $(u,q) \sim (ug,q)$ for $g \in S^1$ and $(u,1) \sim$ a point for $u \in S^{2n-1}$. The generalized reflection

$$j'=j'_n: EP_+^{n-1} \longrightarrow U(n)$$

is defined as the mapping induced from ϕ.

Let $e_1=(1,0,\cdots,0) \in S^{2n-1}$ and take $[e_1] \in P^{n-1}$ as the base point of P^{n-1}. Then we define

$$j=j_n: EP^{n-1} \longrightarrow SU(n)$$

by

$$j([u] \wedge q)=j'([u,q]) \begin{bmatrix} q^{-1} & & 0 \\ & 1 & \\ & & \ddots & \\ 0 & & & 1 \end{bmatrix},$$

where $[u] \wedge q \in P^{n-1} \wedge S^1=EP^{n-1}$. Obviously, j' and j are homeomorphisms into.

According to [17], $SU(n)$ is a cell complex composed of 2^{n-1} cells: 0-dim. cell e^0 and $\{2(k_1+k_2+ \cdots +k_m)-n\}$-dim. cells $e^{2k_1-1,2k_2-1,\cdots,2k_m-1}$ for $n \geq k_1 \geq k_2 \geq \cdots \geq k_m \geq 2$ and $m \geq 1$. In particular, e^{2k-1} is identified with the $(2k-1)$-dim. cell of EP^{n-1} by j_n, where $k=2, \cdots, n$.

The S^1-transfer map

$$g_n : E^{2n+1}P^{n-1} \longrightarrow S^{2n}$$

is defined by taking the adjoint of the composition

$$EP^{n-1} \xrightarrow{j} SU(n) \subset U(n) \xrightarrow{r} SO(2n) \xrightarrow{i} \Omega^{2n}S^{2n},$$

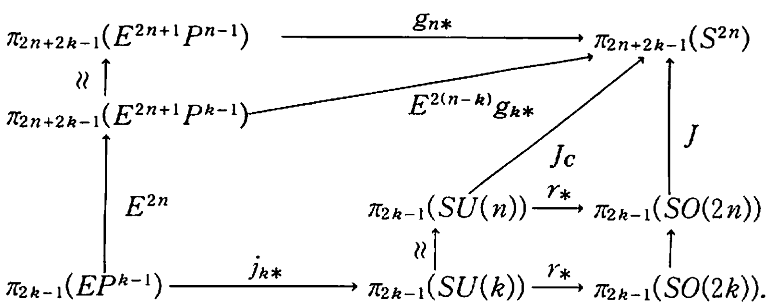
where r is the realization and i is the canonical inclusion.

It follows from the definition that

$$(1) \qquad g_n \,|\, E^{2n+1}P^{n-2} = E^2 g_{n-1}.$$

Let $J: \pi_r(SO(n)) \longrightarrow \pi_{n+r}(S^n)$ (resp. $J_C: \pi_r(SU(n)) \longrightarrow \pi_{2n+r}(S^{2n})$) be the J homomorphism (resp. complex J homomorphism). Then it is easy to prove

Proposition 2.1. *The following diagram is commutative for* $k \leq n$:



We recall that $\pi_k(SU(n))$ is isomorphic to $\boldsymbol{Z}$ (resp. 0) for $k < 2n$ if k is odd (resp. even) by [4].

3. The characteristic map of a unitary bundle. We regard S^{2n} as a subspace of S^{2n+1} consisting of points $z=(z_0,\cdots,z_n)$ such that $\mathrm{Re}(z_n)=0$ and $\sum_{k=0}^{n}|z_k|^2=1$. According to § 24.2 of [11], the characteristic map $T'_n: S^{2n} \longrightarrow U(n)$ for the normal form of a unitary bundle is given by

$$T'_n(z)=(\delta_{ij}-2\bar{z}_i z_j/(1+z_n)^2), \ 0 \leq i, \ j \leq n-1.$$

Put $u_{k+1}=z_k/\sqrt{1+z_n^2}$ for $0 \leq k \leq n-1$, where $z_n \neq \pm\sqrt{-1}$. Put $u=(u_1,\cdots,u_n) \in S^{2n-1}$ and $q=-(1-z_n)^2/(1+z_n)^2 \in S^1$. Then it follows immediately that $T'_n(z)=j'_n([u,q])$. So we have the following

Lemma 3.1. *The characteristic map* $T'_n: S^{2n} \longrightarrow U(n)$ *is homotopic to the composition*

$$S^{2n} \xrightarrow{\pm E\pi_{n-1}} EP^{n-1} \xrightarrow{j_n} SU(n) \subset U(n).$$

From now on, we use often the same letter for a map and its homotopy class.

As is well known, $\pi_{2n}(SU(n)) \approx Z_{n!}$ for $n \geq 2$. So Theorem 25.2 of [11] and Lemma 3.1 lead us to the following

Proposition 3.2. *$j_n E\pi_{n-1}$ generates $\pi_{2n}(SU(n)) \approx Z_{n!}$ and $j_* : \pi_{2n}(EP^{n-1}) \longrightarrow \pi_{2n}(SU(n))$ is an epimorphism for $n \geq 2$.*

4. The Toda map $\zeta^{m,n}$. According to [13] and [14], there exists a cellular mapping $\zeta : E^3P^\infty \longrightarrow EP^\infty$ such that the degree of

$$\zeta_* : H_{2i+1}(E^3P^\infty) \longrightarrow H_{2i+1}(EP^\infty)$$

is i for $i \geq 2$. We define a mapping

$$\zeta^{m,n} : E^{2m+1}P^n \longrightarrow EP^{m+n}$$

as the composition $\zeta \circ E^2\zeta \circ \cdots \circ E^{2(m-1)}\zeta : E^{2m+1}P^n \longrightarrow E^{2m-1}P^{n+1} \longrightarrow \cdots \longrightarrow E^3P^{m+n-1} \longrightarrow EP^{m+n}$. ($\zeta^{0,n}$=the identity map). In particular, we denote

$$\zeta_n = \zeta^{n-1,1} : S^{2n+1} \longrightarrow EP^n.$$

Hideyuki Kachi pointed the author out the following [13]:

Theorem 4.1 (Toda). *$j_{n+1}\zeta_n$ generates $\pi_{2n+1}(SU(n+1)) \approx Z$ and $j_* : \pi_{2n+1}(EP^n) \longrightarrow \pi_{2n+1}(SU(n+1))$ is a split epimorphism for $n \geq 1$.*

By Proposition 3.2 and Theorem 4.1, we have a split exact sequence for $n \geq 1$:

$$(2) \quad 0 \rightarrow \pi_{2n+2}(SU(n+1), EP^n) \xrightarrow{\delta} \pi_{2n+1}(EP^n) \xrightarrow{j_*} \pi_{2n+1}(SU(n+1)) \rightarrow 0.$$

Proposition 4.2. *Let H be the infinite cyclic subgroup of $\pi_{2n+1}(EP^n)$ generated by ζ_n. Then, the restriction homomorphism*

$$E^k|H : H \longrightarrow \pi_{2n+k+1}(E^{k+1}P^n)$$

is a split monomorphism for $k \geq 1$.

Proof. Let m be an integer such that $k \leq 2m$. Consider the composition of the homomorphisms $j_{m+n+1*}\zeta_*^{m,n}E^{2m} : \pi_{2n+1}(EP^n) \longrightarrow \pi_{2m+2n+1}(SU(m+n+1)) \approx Z$. Then this is a split epimorphism by Theorem 4.1, since $\zeta^{m,n}E^{2m}\zeta_n = \zeta_{m+n}$. This completes the proof.

Professor Hirosi Toda suggested the present form of Proposition 4.2 to the author.

By Theorem 4.1, Propositions 2.1 and 4.2, we have the following

Theorem 4.3. $J_C(\pi_{2k-1}(SU(n)) = \{E^{2(n-k)} g_k \circ E^{2n} \zeta_{k-1}\} \subset B_{2k-1,n}$ *for* $k \leq n$.

Let $p_n : P^n \longrightarrow P^n/P^{n-1} = S^{2n}$ be the collapsing map and let ι_n be the identity class of $\pi_n(S^n) \approx \boldsymbol{Z}$ for $n \geq 1$. From the definition, we have

$$Ep_n \circ \zeta_n = n! \iota_{2n+1} \text{ for } n \geq 1. \tag{3}$$

5. Proof of Theorem 1.2. Let $i_n : P^{n-1} \longrightarrow P^n$ and $i'_n : SU(n-1) \longrightarrow SU(n)$ be the inclusions and let $p'_n : SU(n) \longrightarrow SU(n)/SU(n-1) = S^{2n-1}$ be the projection. Then we have a commutative diagram

$$\begin{array}{ccccc} EP^{n-2} & \xrightarrow{Ei_{n-1}} & EP^{n-1} & \xrightarrow{Ep_{n-1}} & S^{2n-1} \\ \downarrow j_{n-1} & & \downarrow j_n & & \| \\ SU(n-1) & \xrightarrow{i'_n} & SU(n) & \xrightarrow{p'_n} & S^{2n-1}, \end{array} \tag{4}$$

where the upper sequence is the cofibering and the lower is the fibering.

Hereafter we use simply i and p to denote the natural inclusion and the collapsing map respectively, unless otherwise stated.

Now we give a proof of Theorem 1.2. Let $p: (EP^n, EP^{n-1}) \longrightarrow (S^{2n+1}, *)$ be the collapsing map. Then $p_*: \pi_r(EP^n, EP^{n-1}) \longrightarrow \pi_r(S^{2n+1})$ for $r \leq 2n+2$ is an isomorphism by [3]. So we have an exact sequence for $n \geq 2$:

$$\pi_{2n+1}(EP^n) \xrightarrow{Ep_{n*}} \pi_{2n+1}(S^{2n+1}) \xrightarrow{\Delta} \pi_{2n}(EP^{n-1}) \xrightarrow{Ei_{n*}} \pi_{2n}(EP^n),$$

where $\Delta = \delta \circ p_*^{-1} : \pi_{2n+1}(S^{2n+1}) \xrightarrow{\approx} \pi_{2n+1}(EP^n, EP^{n-1}) \longrightarrow \pi_{2n}(EP^{n-1})$.

Obviously, $(Ei_n)_*(E\pi_{n-1}) = 0$, and so $E\pi_{n-1} = \Delta(a\iota_{2n+1})$ for some integer a. By (3), $n! E\pi_{n-1} = a\Delta(Ep_{n*}\zeta_n) = 0$. By Proposition 3.2, $j_{n*}E\pi_{n-1}$ is of order $n!$. This concludes the assertion of Theorem 1.2 for $k=0$. It is noted that $(a, n!) = 1$.

Next assume that $k \geq 1$ and let m be an integer such that $2m \geq k$. We consider homomorphisms between the exact sequences

$$\begin{array}{ccccc} \pi_{2n+1}(EP^n) & \xrightarrow{Ep_{n*}} & \pi_{2n+1}(S^{2n+1}) & \xrightarrow{\Delta} & \pi_{2n}(EP^{n-1}) \\ \downarrow E^{2m} & & \wr\!\wr \downarrow E^{2m} & & \downarrow E^{2m} \\ \pi_r(E^{2m+1}P^n) & \xrightarrow{E^{2m+1}p_{n*}} & \pi_r(S^r) & \xrightarrow{\Delta'} & \pi_{r-1}(E^{2m+1}P^{n-1}), \end{array}$$

where $r = 2m+2n+1$. Using the case that $k=0$, we have

$$E^{2m+1}\pi_{n-1} = a\Delta'(\iota_{2m+2n+1}).$$

So it is sufficient to prove that $\Delta'(\iota_{2m+2n+1})$ is of order $n!$.

Let x be the order of the above element. Assume that $x< n!$ and put $y=n!/x \geq 2$. Then there exists an element $\alpha\in\pi_r(E^{2m+1}P^n)$ such that $E^{2m+1}p_{n*}\alpha=x\iota_r$ for $r=2m+2n+1$. By (3), there exists an element $\beta\in\pi_r(E^{2m+1}P^{n-1})$ such that $E^{2m}\zeta_n=y\alpha+i_*\beta$, where $i=E^{2m+1}i_n$. Consider the commutative diagram

$$\begin{array}{ccccc}
\pi_r(E^{2m+1}P^{n-1}) & \xrightarrow{\zeta_*^{m,n-1}} & \pi_r(EP^{m+n-1}) & \xrightarrow{j_*} & \pi_r(SU(m+n)) \\
\downarrow i_* & & \downarrow i'_* & & \downarrow i''_* \\
\pi_r(E^{2m+1}P^n) & \xrightarrow{\zeta_*^{m,n}} & \pi_r(EP^{m+n}) & \xrightarrow{j_*} & \pi_r(SU(m+n+1)).
\end{array}$$

Then, $j_*\zeta_*^{m,n}i_*\beta=i''_*j_*\zeta_*^{m,n-1}\beta=0$, since $\pi_{2m+2n+1}(SU(m+n))\approx Z_2$ or 0 by Theorem 4.4 of [13]. Therefore, $j_*\zeta_{m-n}=j_*\zeta_*^{m,n}E^{2m}\zeta_n=yj_*\zeta_*^{m,n}\alpha$. This contradicts Theorem 4.1. Hence $\Delta'(\iota_{2m+2n+1})$ is of order $n!$. This completes the proof of Theorem 1.2.

6. A characterization of ζ_n and $E\pi_n$. We shall prepare a lemma concerning the Toda bracket. Let X, Y and Z be spaces with base points. Let $[X,Y]$ be the set of homotopy classes of base point preserving maps from X to Y. We denote by $W=Z\cup_\alpha CY$ the mapping cone of $\alpha\in[Y,Z]$. Let $i: Z\longrightarrow W$ be the inclusion and let $p: W\longrightarrow EY$ be the mapping which shrinks Z to a point. We denote by ι_X the homotopy class of the identity map of X.

Lemma 6.1. i) $\iota_{EY}\in\{p, i, \alpha\}$ mod $p_*[EY,W]+(E\alpha)^*[EZ,EY]$ *if* $Y=EY'$.

ii) *Assume that* $0\notin\{p, i, \alpha\}$ *and* $\alpha\beta=0$, *where* $\beta\in[X,Y]$ *and* $X=EX'$. *Suppose given an element* $\tilde\beta\in[EX,W]$ *such that* $p_*\tilde\beta=-E\beta$. *Then*

a) $\tilde\beta\in\{i, \alpha, \beta\}$ mod Ker $p_*+(E\beta)^*[EY,W]$.

Furthermore assume that the sequence

$$[EX,Z]\xrightarrow{i_*}[EX,W]\xrightarrow{p_*}[EX,EY]$$

is exact. Then

b) $\tilde\beta\in\{i, \alpha, \beta\}$ mod $i_*[EX,Z]+(E\beta)^*[EY,W]$.

Proof. By (1.14) and Lemma 1.1 of [15], we have i).

By Proposition 1.4 of [15], $p_*\{i, \alpha, \beta\}=-\{p, i, \alpha\}\circ E\beta$. So, by i) and the assumption, there exists an element $\gamma\in\{i, \alpha, \beta\}$ such that $p_*\gamma=-E\beta$.

Therefore, $\tilde{\beta}-\gamma \in \operatorname{Ker} p_*$. By the definition, $\{i, \alpha, \beta\}$ is a double coset of $i_*[EX,Z] \subset \operatorname{Ker} p_*$ and $(E\beta)^*[EY,W]$. This completes the proof.

Lemma 6.2. $\iota_{2n+1} \in \{Ep_n, Ei_n, E\pi_{n-1}\}$ mod $n!\iota_{2n+1}$ *for* $n \geq 2$.

Proof. The bracket is a coset of the subgroup

$$Ep_{n*}\pi_{2n+1}(EP^n)+(E^2\pi_{n-1})^*[E^2P^{n-1},S^{2n+1}].$$

By (3), (4) and Theorem 4.1, $Ep_{n*}\pi_{2n+1}(EP^n)=\{n!\iota_{2n+1}\}$. On the other hand, $[E^2P^{n-1},S^{2n+1}] \approx 0$. This completes the proof.

By Theorem 1.2, (3), Lemmas 6.1 and 6.2, we have the following

Proposition 6.3. *For* $n \geq 2$,

$\zeta_n \in -\{Ei_n, E\pi_{n-1}, n!\iota_{2n}\}$ mod $Ei_{n*}\pi_{2n+1}(EP^{n-1})+n!\pi_{2n+1}(EP^n)$.

Hereafter we use the following [15]:

$$\pi_{n+1}(S^n)=\{\eta_n\} \approx \boldsymbol{Z}_2 \text{ (resp. } \boldsymbol{Z}) \text{ for } n \geq 3 \text{ (resp. } n=2).$$

By Theorem 24.3 of [11], (4) and Proposition 3.2, we have the following

Lemma 6.4. $p_n\pi_n = n\eta_{2n}$ *for* $n \geq 1$.

Proposition 6.5. i) *For odd* $n \geq 3$,

$E\pi_n \in \{Ei_n, E\pi_{n-1}, \eta_{2n}\}$ mod $Ei_{n*}\pi_{2n+2}(EP^{n-1})+\pi_{2n+1}(EP^n)\circ\eta_{2n+1}$.

ii) *There exists an element* $\lambda_n \in \pi_{2n+2}(EP^{n-1})$ *such that*

$$Ei_n\circ\lambda_n = E\pi_n \ (\textit{resp. } 2E\pi_n)$$

for even (resp. odd) $n \geq 2$.

Proof. By Lemma 6.4, $E\pi_{n-1}\circ\eta_{2n} = E\pi_{n-1}\circ(p_n\pi_n) = (E\pi_{n-1}\circ p_n)\circ\pi_n = 0$ for odd $n \geq 3$. Obviously, the sequence

$$\pi_r(EP^{n-1}) \xrightarrow{Ei_{n*}} \pi_r(EP^n) \xrightarrow{Ep_{n*}} \pi_r(S^{2n+1})$$

is exact for $r=2n+2$. So we have i) by Lemmas 6.1 and 6.2. We have also ii) by Lemma 6.4 and the above exact sequence. This completes the proof.

7. Generators of $\pi_{2n+k}(SU(n))$ for $1 \leq k \leq 3$. We consider the exact sequence

(5) $\pi_{r+1}(S^{2n+1}) \xrightarrow{\Delta'} \pi_r(SU(n)) \xrightarrow{i'_*} \pi_r(SU(n+1)) \xrightarrow{p'_*} \pi_r(S^{2n+1}) \xrightarrow{\Delta'} \pi_{r-1}(SU(n))$.

By (2.2) of [9], $\Delta'(\alpha\circ E\beta)=\Delta'(\alpha)\circ\beta$ for $\alpha\in\pi_{t+1}(S^{2n+1})$ and $\beta\in\pi_q(S^t)$, where $2n\leq t\leq q$ and $q=r$ or $r-1$. Using the proof of Theorem 1.2 for $k=0$, we have an integer b with $(b,n!)=1$ such that

(5)′ $$\Delta'(\iota_{2n+1})=bj_nE\pi_{n-1}.$$

Using [4], Theorem 4.4 of [13], (5) and (5)′ for $r=2n+1$, we see that $\Delta': \pi_{2n+2}(S^{2n+1})\longrightarrow\pi_{2n+1}(SU(n))$ is an epimorphism and $\Delta'(\eta_{2n+1})=j_n\circ E\pi_{n-1}\circ\eta_{2n}$. This leads us to the following

Proposition 7.1. i) *$\pi_{2n+1}(SU(n))=\{j_n\circ E\pi_{n-1}\circ\eta_{2n}\}\approx \boldsymbol{Z}_2$ (resp. 0) for even (resp. odd) $n\geq 2$.*

ii) *$j_*: \pi_{2n+1}(EP^{n-1})\longrightarrow\pi_{2n+1}(SU(n))$ is an epimorphism for $n\geq 2$.*

By Theorem 1.2, Propositions 3.2 and 7.1, we have a split exact sequence for $n\geq 2$:

(6) $$0\longrightarrow\pi_{2n+1}(SU(n),EP^{n-1})\xrightarrow{\delta}\pi_{2n}(EP^{n-1})\xrightarrow{j_*}\pi_{2n}(SU(n))\longrightarrow 0.$$

Hereafter we use the following ([13] and [15]): $\pi_{2n+2}(SU(n))\approx \boldsymbol{Z}_{(n+1)!}+\boldsymbol{Z}_2$ (resp. $\boldsymbol{Z}_{(n+1)!/2}$) for even (resp. odd) $n\geq 3$, $\pi_{n+2}(S^n)=\{\eta_n^2\}\approx \boldsymbol{Z}_2$ for $n\geq 2$ and $\pi_6(S^3)\approx \boldsymbol{Z}_{12}$.

Proposition 7.2. i) *$j_{n*}: \pi_{2n+2}(EP^{n-1})\longrightarrow\pi_{2n+2}(SU(n))$ is an epimorphism for $n\geq 2$.*

ii) *$\pi_{2n+2}(SU(n))=\{j_n\lambda_n, j_n\circ E\pi_{n-1}\circ\eta_{2n}^2\}\approx \boldsymbol{Z}_{(n+1)!}+\boldsymbol{Z}_2$ for even $n\geq 4$ and $\pi_6(SU(2))=\{j_2\lambda_2\}\approx \boldsymbol{Z}_{12}$.*

iii) *$\pi_{2n+2}(SU(n))=\{j_n\lambda_n\}\approx \boldsymbol{Z}_{(n+1)!/2}$ for odd $n\geq 3$.*

Proof. By inspecting the proof of Theorem 4.3 of [13], we have a commutative diagram for $n\geq 3$:

$$\begin{array}{ccc}
\pi_{2n+3}(EP^{n+1},EP^{n-1}) & \xrightarrow{\delta} & \pi_{2n+2}(EP^{n-1}) \\
\downarrow p_* & & \\
\pi_{2n+3}(EP^{n+1}/EP^{n-1}) & & \downarrow j_{n*} \\
\wr\downarrow & & \\
\pi_{2n+3}(SU(n+2)/SU(n)) & \xrightarrow{\Delta'} & \pi_{2n+2}(SU(n)).
\end{array}$$

By [4], $\pi_{2n+2}(SU(n+2))\approx 0$, and so Δ' is an epimorphism. By [3], p_* is an epimorphism. This leads us to i).

Using Proposition 7.1. i), (5) and (5)′ for $r=2n+2$, we see that

$$i'_*: \pi_{2n+2}(SU(n)) \longrightarrow \pi_{2n+2}(SU(n+1))$$

is an epimorphism for even $n \geq 2$. By considering the group structure of $\pi_{2n+2}(SU(k))$ for $k=n$ and $n+1$, $\Delta'(\eta^2_{2n+1})=j_n \circ E\pi_{n-1} \circ \eta^2_{2n} \neq 0$ (resp. $=0$) for even (resp. odd) $n \geq 3$. So, i'_* is a monomorphism for odd $n \geq 3$. Hence (4), Propositions 3.2 and 6.5. ii) lead us to ii) and iii). This completes the proof.

By Propositions 7.1 and 7.2, we have a split exact sequence for even $n \geq 2$:

(7) $$0 \longrightarrow \pi_{2n+2}(SU(n),EP^{n-1}) \xrightarrow{\delta} \pi_{2n+1}(EP^{n-1}) \xrightarrow{j_*} \pi_{2n+1}(SU(n)) \longrightarrow 0.$$

We have also an isomorphism for odd $n \geq 3$:

(7)′ $$\delta: \pi_{2n+2}(SU(n),EP^{n-1}) \xrightarrow{\approx} \pi_{2n+1}(EP^{n-1}).$$

From now on, we use the same symbol to denote generators of $\pi_{n+k}(S^n)$ and its 2-primary component.

Hereafter we use the following ([8] and [15]): $\pi_{2n+3}(SU(n)) \approx Z_{a(n)}$, where $a(n)=(24,n)$ (resp. $(24,n+3)/2$) for even (resp. odd) $n \geq 2$. $\pi_6(S^3)=\{\nu'\} \approx Z_{12}$, $\pi_7(S^4)=\{\nu_4, E\nu'\} \approx Z+Z_{12}$ and $\pi_{n+3}(S^n)=\{\nu_n\} \approx Z_{24}$ for $n \geq 5$. $6\nu'=\eta^3_3$ and $2\nu_n=E^{n-3}\nu'$ for $n \geq 5$.

Proposition 7.3. i) *$\pi_{2n+3}(SU(n))=\{j_n \circ E\pi_{n-1} \circ \nu_{2n}\} \approx Z_{a(n)}$ for $n \geq 2$.*
ii) *$j_*: \pi_{2n+3}(EP^{n-1}) \longrightarrow \pi_{2n+3}(SU(n))$ is an epimorphism for $n \geq 2$.*

Proof. We consider the exact sequence (5) for $r=2n+3$:

$$\pi_{r+1}(S^{2n+1}) \xrightarrow{\Delta'} \pi_r(SU(n)) \xrightarrow{i'_*} \pi_r(SU(n+1)) \xrightarrow{p'_*} \pi_r(S^{2n+1}).$$

By Proposition 7.1. i), (4) and Lemma 6.4, p'_* is a monomorphism for $n \geq 1$. So Δ' is an epimorphism for $n \geq 1$. By (5)′, $\Delta'(\nu_{2n+1})=b(j_n \circ E\pi_{n-1} \circ \nu_{2n})$ for $n \geq 2$. This completes the proof.

By Propositions 7.2 and 7.3, we have a short exact sequence for $n \geq 2$:

(8) $$0 \longrightarrow \pi_{2n+3}(SU(n),EP^{n-1}) \xrightarrow{\delta} \pi_{2n+2}(EP^{n-1}) \xrightarrow{j_*} \pi_{2n+2}(SU(n)) \longrightarrow 0.$$

8. **Determination of $\pi_{n+k}(E^{n+1}P^m)$ for $k \leq 7$.** We consider an exact sequence

$$\pi_{n+k+1}(E^{n+1}P^m, E^{n+1}P^{m-1}) \xrightarrow{\delta} \pi_{n+k}(E^{n+1}P^{m-1}) \xrightarrow{i_*} \pi_{n+k}(E^{n+1}P^m) \xrightarrow{j_*} \pi_{n+k}(E^{n+1}P^m, E^{n+1}P^{m-1}).$$

By [3], $p'_*: \pi_{n+k+1}(E^{n+1}P^m, E^{n+1}P^{m-1}) \longrightarrow \pi_{n+k+1}(S^{n+2m+1})$ is an isomorphism for $n \geq k-2m-1$ and $m \geq 2$. So we have the exact sequence for $n \geq k-2m-1$ and $m \geq 2$:

$(9)_{m,k}$ $$\pi_{n+k+1}(S^{n+2m+1}) \xrightarrow{\Delta} \pi_{n+k}(E^{n+1}P^{m-1}) \xrightarrow{i_*} \pi_{n+k}(E^{n+1}P^m) \xrightarrow{p_*} \pi_{n+k}(S^{n+2m+1}),$$

where $\Delta = \Delta_m = \delta \circ p'^{-1}_*$.

We have also the exact sequence for $n \geq k-2m-2$ and $m \geq 2$:

$(9)'_{m,k}$ $$\pi_{n+k}(E^{n+1}P^{m-1}) \xrightarrow{i_*} \pi_{n+k}(E^{n+1}P^m) \xrightarrow{p_*} \pi_{n+k}(S^{n+2m+1}) \xrightarrow{\Delta} \pi_{n+k-1}(E^{n+1}P^{m-1}).$$

As is well known, $\Delta(\alpha \circ E\beta) = \Delta(\alpha) \circ \beta$, where $\alpha \in \pi_{t+1}(S^{n+2m+1})$ and $\beta \in \pi_{n+k}(S^t)$ for $n \geq k-2m-1$ and $m \geq 2$. By inspecting the proof of Theorem 1.2, $\Delta(\iota_{n+2m+1}) = bE^{n+1}\pi_{m-1}$, and so $\Delta(E\alpha) = b(E^{n+1}\pi_{m-1} \circ \alpha)$ for $\alpha \in \pi_{n+k}(S^{n+2m})$, where b is an integer such that $(b, n!) = 1$. For the simplicity, we use the following expression, because it makes no difference to continuation of the subsequent arguments.

$(10)_m$ $\quad \Delta_m(E\alpha) = E^{n+1}\pi_{m-1} \circ \alpha$ for $n \geq k-2m-1$ and $m \geq 2$.

Now, we start to compute $\pi_{n+k}(E^{n+1}P^m)$.

By Lemma 6.4, we have

(11) $$\pi_1 = \eta_2.$$

By $(10)_2$ and (11), we have

(12) $\quad \Delta_2(\iota_{n+5}) = \eta_{n+3}$ and $\Delta_2(\eta_{n+5}) = \eta^2_{n+3}$ for $n \geq 0$.

Using (3), (12) and $(9)_{2,k}$ for $k=4$ and 5, we have the following

Proposition 8.1. i) $\pi_{n+k}(E^{n+1}P^1) \approx 0$ *for* $k \leq 2$ *and* $n \geq 0$.
ii) $\pi_{n+3}(E^{n+1}P^1) = \{E^n\zeta_1\} \approx \boldsymbol{Z}$ *for* $n \geq 0$.
iii) $\pi_{n+4}(E^{n+1}P^2) \approx 0$ *for* $n \geq 0$.
iv) $\pi_{n+5}(E^{n+1}P^2) = \{E^n\zeta_2\} \approx \boldsymbol{Z}$ *for* $n \geq 0$.

By [17], $SU(3) = EP^2 \cup e^{5,3}$, and so $j_{3*}: \pi_k(EP^2) \longrightarrow \pi_k(SU(3))$ is an isomorphism for $k \leq 6$. Therefore, by Proposition 3.2, $\pi_6(EP^2) = \{E\pi_2\} \approx \boldsymbol{Z}_6$. By $(9)'_{2,6}$ for $n=0$ and by (12), $i_*: \pi_6(S^3) \longrightarrow \pi_6(EP^2)$ is an epimorphism, where $i=Ei_2: S^3 \longrightarrow EP^2$. By (11), $i\eta_3=0$. Hence we have

(13) $$E\pi_2=\pm i\nu'.$$

By $(10)_2$ and (12), we have

(14) $$\Delta_2(\eta^2_{n+5})=12\nu_{n+3} \text{ (resp. } 6E\nu') \text{ for } n \geq 2 \text{ (resp. } n=1).$$

Using $(9)_{2,6}$, $(9)'_{2,6}$, (14) and the above argument, we have the following

Proposition 8.2. i) $\pi_6(EP^2)=\{E\pi_2\} \approx Z_6$.
ii) $\pi_7(E^2P^2)=\{i\nu_4, iE\nu'\} \approx Z+Z_6$.
iii) $\pi_{n+6}(E^{n+1}P^2)=\{i\nu_{n+3}\} \approx Z_{12}$ *for* $n \geq 2$.

Proposition 8.3. *Let* ξ *be a generator of* $\pi_7(EP^2)$, *and* $i'=Ei_3: EP^2 \longrightarrow EP^3$.
i) $\pi_7(EP^2)=\{\xi\} \approx Z$.
ii) $\pi_7(EP^3)=\{i'\xi, \zeta_3\} \approx Z+Z$.

Proof. By [3] and [17], $\pi_8(SU(k), EP^{k-1}) \approx \pi_8(S^8) \approx Z$ for $k \geq 3$. So, by (2) and (7)' for $n=3$, we have the assertion.

Hereafter we use the following [15] : $\pi_8(S^4)=\{\nu_4\eta_7, E\nu'\eta_7\} \approx Z_2+Z_2$, $\pi_9(S^5)=\{\nu_5\eta_8\} \approx Z_2$ and $\pi_{n+4}(S^n) \approx 0$ for $n \geq 6$. $\nu'\eta_6=\eta_3\nu_4$, $\eta_n\nu_{n+1}=0$ and $\nu_{n+1}\eta_{n+4}=0$ for $n \geq 5$.

By $(10)_2$ and (11), $\Delta_2(\nu_{n+5})=\eta_{n+3}\nu_{n+4}=0$ for $n \geq 2$. So we have

(15) $$\Delta_2(\nu_{n+5})=0 \text{ for } n \geq 2.$$

By $(9)_{2,7}$, $(9)'_{2,7}$, (14) and (15), we have the following

Proposition 8.4. i) $\pi_{n+7}(E^{n+1}P^2)=\{i\nu_{n+3}\eta_{n+6}\} \approx Z_2$ *for* $n=0$ *or* 1.
ii) $\pi_{n+7}(E^{n+1}P^2) \approx 0$ *for* $n \geq 3$.

Lemma 8.5. $E\xi=i\nu_4\eta_7$.

Proof. Using the EHP-sequence (cf. Theorem 2.2 of [16, Chap. 12]), we have the exact sequence

$$\pi_7(EP^2) \xrightarrow{E} \pi_8(E^2P^2) \xrightarrow{H} \pi_8(EP^2 * EP^2) \xrightarrow{P} \pi_6(EP^2),$$

where $EP^2 * EP^2$ denotes the reduced join.

Clearly, $\pi_8(EP^2 * EP^2) \approx \pi_8(E^3(P^2 \wedge P^2)) \approx \pi_8(E^5P^2 \vee S^9) \approx \pi_8(E^5P^2) \approx 0$ by Proposition 8.1. iii). So E is an epimorphism. Hence we have the assertion by Propositions 8.3 and 8.4.

Lemma 8.6. i) $\Delta_3(\iota_{n+7})=\pm 2i\nu_{n+3}$ *(resp.* $\pm iE^n\nu'$*) for* $n\geq 2$ *(resp.* $n=0$ *or* 1).

ii) $\Delta_3(\eta_{n+7})=0$ *for* $n\geq 0$.

Proof. By $(10)_3$ and (13), we have i). So, $\Delta_3(\eta_{n+7})=2i\nu_{n+3}\eta_{n+6}=0$ for $n\geq 2$ and $\Delta_3(\eta_{n+7})=iE^n(\nu'\eta_6)=iE^n(\eta_3\nu_4)=0$ for $n=0$ or 1. This completes the proof.

Using $(9)_{3,6}$, $(9)_{3,7}$, Lemma 8.6, (3), Propositions 8.2 and 8.4, we have the following

Proposition 8.7. i) $\pi_6(EP^3)\approx 0$.

ii) $\pi_7(E^2P^3)=\{i\nu_4\}\approx \boldsymbol{Z}$.

iii) $\pi_{n+6}(E^{n+1}P^3)=\{i\nu_{n+3}\}\approx \boldsymbol{Z}_2$ *for* $n\geq 2$.

iv) $\pi_{n+7}(E^{n+1}P^3)=\{E^n\zeta_3,\ i\nu_{n+3}\eta_{n+6}\}\approx \boldsymbol{Z}+\boldsymbol{Z}_2$ *for* $n=1$ *or* 2.

v) $\pi_{n+7}(E^{n+1}P^3)=\{E^n\zeta_3\}\approx \boldsymbol{Z}$ *for* $n\geq 3$.

9. Determination of $\pi_8(EP^m)$ for $m=2$ and 3. In this section we use the following [15]: $3\nu'\in\{\eta_3,\ 2\iota_4,\ \eta_4\}$ mod $6\nu'$, $\pi_7(S^3)=\{\nu'\eta_6\}\approx \boldsymbol{Z}_2$ and $\pi_8(S^3)=\{\nu'\eta_6^2\}\approx \boldsymbol{Z}_2$.

By (11), Propositions 6.3 and 8.1. iv), we have the following

Lemma 9.1. $\zeta_2\in\{i,\ \eta_3,\ 2\iota_4\}$ mod $2\zeta_2$.

Proposition 9.2. i) $\lambda_3\equiv\pm\zeta_2\nu_5$ mod $\xi\eta_7$.

ii) $\pi_8(EP^2)=\{\zeta_2\nu_5,\ \xi\eta_7\}\approx \boldsymbol{Z}_{12}+\boldsymbol{Z}_2$.

Proof. By (4), we have the commutative diagram

$$\begin{array}{ccccccc} & \overset{i_*}{\nearrow} & \pi_8(EP^2) & \overset{p_*}{\searrow} & & & \\ & & \downarrow{\scriptstyle j_*} & & & & \\ \pi_8(S^3) & \xrightarrow{i'_*} & \pi_8(SU(3)) & \xrightarrow{p'_*} & \pi_8(S^5) & \xrightarrow{\Delta'} & \pi_7(S^3), \end{array}$$

where the horizontal sequence is exact.

Since $i'_*\pi_8(S^3)=\{j_*(i_*\nu'\eta_6^2)\}=0$, p'_* is a monomorphism. Therefore, Ker p_*=Ker j_*. By $(5)'$ and (11), $\Delta'(\nu_5)=\eta_3\nu_4$, and so Im $p'_*=\{2\nu_5\}$. Therefore, by (3) and Proposition 7.2,

$$p_*(\zeta_2\nu_5)=2\nu_5=p_*(\pm\lambda_3).$$

On the other hand, we consider the exact sequence (8) for $n=3$:

$$0\longrightarrow \pi_9(SU(3),EP^2)\xrightarrow{\delta}\pi_8(EP^2)\xrightarrow{j_*}\pi_8(SU(3))\longrightarrow 0.$$

By [3] and [17], $\pi_9(SU(3), EP^2) \approx \pi_9(S^8) \approx Z_2$. By inspecting the proof of Proposition 8.3, we obtain that Im $\delta=\{\xi\eta_7\}$. This leads us to i).

By Proposition 1.4 of [15] and by Lemma 9.1, $12\zeta_2\nu_5=\zeta_2\eta_5^3 \in \{i, \eta_3, 2\iota_4\}\circ\eta_5^3=-i\{\eta_3, 2\iota_4, \eta_4\}\circ\eta_6^2=i\nu'\eta_6^2=0$ mod $2\zeta_2\eta_5^3=0$. So the order of $\zeta_2\nu_5$ is 12 by i) and by Proposition 7.2. iii). This completes the proof.

Lemma 9.3. i) $\zeta_3 \in \pm\{i', i\nu', 6\iota_6\}$ mod $\{i'\xi, 6\zeta_3\}$.
ii) $E\pi_3 \in \{i', i\nu', \eta_6\}$ mod $\{i'\zeta_2\nu_5, i'\xi\eta_7\}$.

Proof. By Proposition 1.2 of [15] and by (13), Propositions 6.3 and 8.3, we have i).

By Proposition 1.4 of [15] and by i), $\zeta_3\eta_7 \in \{i', i\nu', 6\iota_6\}\circ\eta_7= -i'\{i\nu', 6\iota_6, \eta_6\} \subset i'_*\pi_8(EP^2)$. By Proposition 1.2 of [15], (13) and Proposition 6.5. i), $E\pi_3 \in \{i', \pm i\nu', \eta_6\} \supset \{i', i\nu', \eta_6\}$ mod $i'_*\pi_8(EP^2)+\pi_7(EP^3)\circ\eta_7$. So we have ii) by Propositions 8.3. ii) and 9.2. ii). This completes the proof.

To determine the group structure of $\pi_8(EP^3)$, we need the following

Lemma 9.4. *The* 10*-skeleton of the complex* $SU(n)/EP^{n-1}$ *for* $n \geq 4$ *is* $S^8 \vee S^{10}$.

Proof. As is well known, the cohomology ring $H^*(SU(n); Z_2)$ is isomorphic to the exterior algebra $E_{Z_2}[x_2, \cdots, x_n]$, where deg. $x_k=2k-1$. By abuse of notation, we denote by $e^3, e^5, \cdots, e^{2n-1}$ the cohomology classes corresponding to the generators $x_2, x_3, \cdots, x_n$.

Let $Sq^2: H^*(X;Z_2) \longrightarrow H^*(X;Z_2)$ be the squaring operation [12]. Since the 7-skeleton of $SU(n)$ is EP^3, $Sq^2e^3=e^5$ and $Sq^2e^5=0$. By the Cartan formula, $Sq^2(e^3e^5)=(e^5)^2=0$. This completes the proof.

Proposition 9.5. i) $2E\pi_3 \equiv \pm i'\zeta_2\nu_5$ mod $i'\xi\eta_7$.
ii) $\pi_8(EP^3)=\{E\pi_3, i'\xi\eta_7\} \approx Z_{24}+Z_2$.

Proof. By [3] and Lemma 9.4, $\pi_9(SU(4), EP^3) \approx \pi_9(SU(4)/EP^3) \approx \pi_9(S^8 \vee S^{10}) \approx Z_2$. So, by (6) for $n=4$, $\pi_8(EP^3) \approx Z_{24}+Z_2$. Hence, by Theorem 1.2, Propositions 6.5. ii) and 9.2, we have the assertion.

10. Determination of $\pi_{n+k}(E^{n+1}P^m)$ for $k=8, 9$ and $n \geq 3$. Hereafter we use the following [15]: $\pi_{10}(S^5)=\{\nu_5\eta_8^2\} \approx Z_2$, $\pi_{11}(S^6)=\{[\iota_6, \iota_6]\} \approx Z$ and $\pi_{n+5}(S^n) \approx 0$ for $n \geq 7$, where $[\iota_6, \iota_6]$ denotes the Whitehead product.

We consider the exact sequence $(9)'_{2,8}$ for $n \geq 2$:

$$\pi_{n+8}(S^{n+3}) \xrightarrow{i_*} \pi_{n+8}(E^{n+1}P^2) \xrightarrow{p_*} \pi_{n+8}(S^{n+5}) \xrightarrow{\Delta} \pi_{n+7}(S^{n+3}).$$

By (15), p_* is an epimorphism. So there exists an element $\alpha \in \pi_{10}(E^3P^2)$ such that $p_*\alpha = -\nu_7$. We write $\tilde{\nu}_{n+7} = E^n\alpha$ for $n \geq 0$. By Lemmas 6.1, 6.2 and Proposition 8.1. iii), we have

$$\begin{aligned} &p_*\tilde{\nu}_{n+5} = -\nu_{n+5} \text{ for } n \geq 2, \\ (16) \qquad &\tilde{\nu}_7 \in \{i, \eta_5, \nu_6\} \bmod \{i\nu_5\eta_8^2, E^2(\zeta_2\nu_5)\}, \text{ and} \\ &\tilde{\nu}_{n+5} \in \{i, \eta_{n+3}, \nu_{n+4}\} \bmod E^n(\zeta_2\nu_5) \text{ for } n \geq 4. \end{aligned}$$

Lemma 10.1. *$\tilde{\nu}_n$ is of order 24 for $n \geq 7$.*

Proof. It is sufficient to show that $24\tilde{\nu}_7 = 0$. By Proposition 1.4 of [15] and by (16), $24\tilde{\nu}_7 \in \{i, \eta_5, \nu_6\}\circ 24\iota_{10} = -i\{\eta_5, \nu_6, 24\iota_9\}$. By Proposition 1.2 of [15], $\{\eta_5, \nu_6, 24\iota_9\} \subset \{\eta_5, 12\nu_6, 2\iota_9\} = \{\eta_5, \eta_6^3, 2\iota_9\} \supset \{\eta_5^3, \eta_8, 2\iota_9\} = \{12\nu_5, \eta_8, 2\iota_9\} \supset 2\{6\nu_5, \eta_8, 2\iota_9\} \subset 2\pi_{10}(S^5) = 0$. So we have $\{\eta_5, \nu_6, 24\iota_9\} \ni 0 \bmod \eta_5\circ\pi_{10}(S^6) + 2\pi_{10}(S^5) = 0$. Therefore $24\tilde{\nu}_7 = 0$. This completes the proof.

By $(9)_{2,8}$, (16) and Lemma 10.1, we have the following

Proposition 10.2. i) $\pi_{11}(E^4P^2) = \{i[\iota_6, \iota_6], \tilde{\nu}_8\} \approx \boldsymbol{Z} + \boldsymbol{Z}_{24}$.
ii) $\pi_{n+8}(E^{n+1}P^2) = \{\tilde{\nu}_{n+5}\} \approx \boldsymbol{Z}_{24}$ *for* $n \geq 4$.

Lemma 10.3. i) $2E^3\pi_3 \equiv \pm 2i'\tilde{\nu}_7 \bmod i\nu_5\eta_8^2$.
ii) $2E^{n+1}\pi_3 = \pm 2i'\tilde{\nu}_{n+5}$ *for* $n \geq 3$.

Proof. By (3) and (16), $p_*(2\tilde{\nu}_7 + E^2(\zeta_2\nu_5)) = 0$. So, by $(9)'_{2,8}$ for $n=2$, $2\tilde{\nu}_7 \equiv -E^2(\zeta_2\nu_5) \bmod i\nu_5\eta_8^2$. Therefore, by Proposition.9.5. i) and Lemma 8.5, we have the assertion.

By Lemma 8.6. ii), we have

$$(17) \qquad \Delta_3(\eta_{n+7}^2) = 0 \text{ for } n \geq 1.$$

By $(9)_{3,8}$, $(9)'_{3,8}$, Lemma 8.6. ii) and (17), we have a short exact sequence for $n \geq 1$:

$$0 \longrightarrow \pi_{n+8}(E^{n+1}P^2) \xrightarrow{i'_*} \pi_{n+8}(E^{n+1}P^3) \xrightarrow{p'_*} \pi_{n+8}(S^{n+7}) \longrightarrow 0,$$

where $p' = E^{n+1}p_3$. So, by Proposition 10.2 and Lemma 10.3, we have the following

Proposition 10.4. i) $\pi_{11}(E^4P^3) = \{i[\iota_6, \iota_6], i'\tilde{\nu}_8, E^4\pi_3\} \approx \boldsymbol{Z} + \boldsymbol{Z}_{24} + \boldsymbol{Z}_2$.
ii) $\pi_{n+8}(E^{n+1}P^3) = \{i'\tilde{\nu}_{n+5}, E^{n+1}\pi_3\} \approx \boldsymbol{Z}_{24} + \boldsymbol{Z}_2$ *for* $n \geq 4$.

Hereafter we use the following [15] : $\pi_{n+6}(S^n)=\{\nu_n^2\}\approx Z_2$ for $n\geq 5$. By $(9)_{2,9}$ and $(9)'_{2,9}$, we have the following

Proposition 10.5. $\pi_{n+9}(E^{n+1}P^2)=\{i\nu_{n+3}^2\}\approx Z_2$ *for* $n\geq 3$.

By Lemma 8.6. i), we have

$$\Delta_3(\nu_{n+7})=0 \text{ for } n\geq 2. \tag{18}$$

By $(9)_{3,9}$, $(9)'_{3,9}$, (17) and (18), we have a short exact sequence for $n\geq 2$:

$$0\longrightarrow \pi_{n+9}(E^{n+1}P^2)\xrightarrow{i'_*}\pi_{n+9}(E^{n+1}P^3)\xrightarrow{p'_*}\pi_{n+9}(S^{n+7})\longrightarrow 0.$$

So, by Lemma 6.4 and Proposition 10.5, we have the following

Proposition 10.6. $\pi_{n+9}(E^{n+1}P^3)=\{i\nu_{n+3}^2,\ E^{n+1}\pi_3\circ\eta_{n+8}\}\approx Z_2+Z_2$ *for* $n\geq 3$.

By $(10)_4$, we have

$$\Delta_4(\iota_{n+9})=E^{n+1}\pi_3 \text{ and } \Delta_4(\eta_{n+9})=E^{n+1}\pi_3\circ\eta_{n+8} \text{ for } n\geq 0. \tag{19}$$

Using $(9)_{4,8}$, $(9)_{4,9}$, (19), Lemma 10.3, (3), Propositions 9.5. ii), 10.4 and 10.6, we have the following

Proposition 10.7. i) $\pi_8(EP^4)=\{i'\xi\eta_7\}\approx Z_2$.

ii) $\pi_{11}(E^4P^4)=\{i[\iota_6,\iota_6],\ i'\tilde{\nu}_8\}\approx Z+Z_2$.

iii) $\pi_{n+8}(E^{n+1}P^4)=\{i'\tilde{\nu}_{n+5}\}\approx Z_2$ *for* $n\geq 4$.

iv) $\pi_{n+9}(E^{n+1}P^4)=\{E^n\zeta_4,\ i\nu_{n+3}^2\}\approx Z+Z_2$ *for* $n\geq 3$.

11. Determination of $\pi_{n+10}(E^{n+1}P^m)$ for $n\geq 4$.

Hereafter we use the following [15]: $\pi_9(S^3)=\{\alpha_1(3)\alpha_1(6)\}\approx Z_3$, where $\alpha_1(3)=4\nu'$ and $\alpha_1(n)=E^{n-3}\alpha_1(3)$ for $n\geq 3$. $\nu'\nu_6=\alpha_1(3)\alpha_1(6)$. $\pi_{10}(S^3)=\{\alpha_2(3),\ \ddot{\alpha}_1(3)\}\approx Z_{15}$, where $\alpha_2(3)$ (resp. $\ddot{\alpha}_1(3)$) denotes the generator of the direct summand Z_3 (resp. Z_5) of $\pi_{10}(S^3)$.

Lemma 11.1. i) $p'_*\lambda_4=\pm 3\nu_7$.

ii) $\lambda_4\in\pm\{i',\ i\nu',\ 3\nu_6\}$ mod $\{3\zeta_3\nu_7\}+\mathrm{Ker}\ p'_*$.

iii) $8\lambda_4\equiv 0$ mod $8(\mathrm{Ker}\ p'_*)$.

Proof. We consider the commutative daiagram

$$\begin{array}{ccccccccc}
 & & \pi_{10}(EP^2) & \xrightarrow{i'_*} & \pi_{10}(EP^3) & \overset{p'_*}{\searrow} & & & \\
 & & \downarrow{j_{3*}} & & \downarrow{j_{4*}} & & & & \\
0 & \longrightarrow & \pi_{10}(SU(3)) & \xrightarrow{i'''_*} & \pi_{10}(SU(4)) & \xrightarrow{p''_*} & \pi_{10}(S^7) & \xrightarrow{\Delta'} & \pi_9(SU(3)),
\end{array}$$

where the horizontal sequence is exact.

By [8], $\pi_9(SU(3)) \approx Z_3$ and $\pi_{10}(SU(3)) \approx Z_{30}$. By the proof of Proposition 7.3, by (5)$'$ and (13), $\Delta'(\nu_7)=\pm j_3 i\nu'\nu_6$ generates $\pi_9(SU(3))$. So, by Proposition 7.2, we have i).

Using Proposition 1.2 of [15], (13), Lemmas 6.1, 6.2 and Proposition 8.3. ii), we have ii).

By Propositions 1.2 and 1.4 of [15], $\pm 8\lambda_4 \in \{i', i\nu', 3\nu_6\}\circ 8\iota_{10} = -i'\{i\nu', 3\nu_6, 8\iota_9\} \supset -i'i\{\nu', 3\nu_6, 8\iota_9\} \subset i_*\pi_{10}(S^3)=8i_*\pi_{10}(S^3) \subset 8(\mathrm{Ker}\ p'_*)$. This leads us to iii). This completes the proof.

Hereafter we use the following [15]: $\pi_{14}(S^7)=\{\sigma'\} \approx Z_{120}$, $\pi_{15}(S^8)=\{\sigma_8, E\sigma'\} \approx Z+Z_{120}$ and $\pi_{n+7}(S^n)=\{\sigma_n\} \approx Z_{240}$ for $n \geq 9$. $40\sigma'=\alpha_2(7)$ and $24\sigma'=\ddot{\alpha}_1(7)$, where $\alpha_2(n)=E^{n-3}\alpha_2(3)$ and $\ddot{\alpha}_1(n)=E^{n-3}\ddot{\alpha}_1(3)$ for $n \geq 3$. $2\sigma_n=E^{n-7}\sigma'$ for $n \geq 9$. $\alpha_2(n) \in 2\{\alpha_1(n), \alpha_1(n+3), 3\iota_{n+6}\}$ for $n \geq 5$ (cf. (13.8) of [15]).

By (15), we have

$$(20)\qquad \Delta_2(\nu_{n+5}^2)=0 \text{ for } n \geq 5.$$

By $(9)_{2,10}$, $(9)'_{2,10}$ and (20), we have the following

Proposition 11.2. i) $\pi_{14}(E^5P^2)=\{i\sigma'\} \approx Z_{120}$.
ii) $\pi_{15}(E^6P^2)=\{i\sigma_8, iE\sigma'\} \approx Z+Z_{120}$.
iii) $\pi_{n+10}(E^{n+1}P^2)=\{i\sigma_{n+3}\} \approx Z_{240}$ *for* $n \geq 6$.

We consider the exact sequence $(9)'_{3,10}$ for $n \geq 2$:

$$\pi_{n+10}(E^{n+1}P^2) \xrightarrow{i'_*} \pi_{n+10}(E^{n+1}P^3) \xrightarrow{p'_*} \pi_{n+10}(S^{n+7}) \xrightarrow{\Delta} \pi_{n+9}(E^{n+1}P^2).$$

By (18), p'_* is an epimorphism. So there exists an element $\beta \in \pi_{12}(E^3P^3)$ such that $p'_*\beta=-\nu_9$. We write $\tilde{\nu}'_{n+9}=E^n\beta$ for $n \geq 0$. Using Proposition 1.2 of [15], Lemmas 6.1, 6.2, (13) and Proposition 8.7, we have

$$(21)\qquad \begin{gathered} p'_*\tilde{\nu}'_{n+7}=-\nu_{n+7}, \text{ and} \\ \tilde{\nu}'_{n+7} \in \pm\{i', 2i\nu_{n+3}, \nu_{n+6}\} \bmod \{E^n(\zeta_3\nu_7)\}+i'_*\pi_{n+10}(E^{n+1}P^2) \text{ for } n \geq 2. \end{gathered}$$

Lemma 11.3. i) $3\tilde{\nu}'_{n+7} \equiv \pm E^n\lambda_4 \bmod iE^{n-4}\sigma'$ *for* $n \geq 4$.
ii) $24\tilde{\nu}'_{n+7} \equiv \pm i\alpha_2(n+3) \bmod i\ddot{\alpha}_1(n+3)$ *for* $n \geq 4$.

Proof. By (21) and Lemma 11.1, $3p'_*\tilde{\nu}'_{11}=p'_*(\pm E^4\lambda_4)=-3\nu_{11}$. So, by Proposition 11.2, $3\tilde{\nu}'_{11}\pm E^4\lambda_4 \in \mathrm{Ker}\ p'_*=i'_*\pi_{14}(E^5P^2)=\{i\sigma'\}$. This leads us to i).

By Proposition 1.4 of [15] and by (21), $\pm 24\tilde{\nu}'_{n+7} \in \{i', 2i\nu_{n+3}, \nu_{n+6}\}$

$\circ 24\iota_{n+10} = -i'\{2i\nu_{n+3}, \nu_{n+6}, 24\iota_{n+9}\}$ mod $24 i'_* \pi_{n+10}(E^{n+1}P^2)$ for $n \geq 2$. By Proposition 1.2 of [15], $\{2i\nu_{n+3}, \nu_{n+6}, 24\iota_{n+9}\} \subset \{2i\nu_{n+3}, 8\nu_{n+6}, 3\iota_{n+9}\} = \{2i\alpha_1(n+3), \alpha_1(n+6), 3\iota_{n+9}\} \supset 2i\{\alpha_1(n+3), \alpha_1(n+6), 3\iota_{n+9}\} \ni i\alpha_2(n+3)$ mod $3\pi_{n+10}(E^{n+1}P^2)$. Therefore, $24\tilde{\nu}'_{n+7} \equiv \pm i\alpha_2(n+3)$ mod $3 i'_* \pi_{n+10}(E^{n+1}P^2)$ for $n \geq 2$. On the other hand, $24\tilde{\nu}'_{11} \equiv 0$ mod $8i\sigma'$ by i), Lemma 11.1. iii) and Proposition 11.2. i). This leads us to ii).

By $(9)_{3,10}$, $(9)'_{3,10}$, Proposition 11.2, (21) and Lemma 11.3, we have the following

Proposition 11.4. i) $\pi_{14}(E^5P^3) = \{i\sigma', \tilde{\nu}'_{11}\} \approx Z_8 + Z_{360}$.
ii) $\pi_{15}(E^6P^3) = \{i\sigma_8, iE\sigma', \tilde{\nu}'_{12}\} \approx Z + Z_8 + Z_{360}$.
iii) $\pi_{n+10}(E^{n+1}P^3) = \{i\sigma_{n+3}, \tilde{\nu}'_{n+7}\} \approx Z_{16} + Z_{360}$ *for* $n \geq 6$.

Lemma 11.5. i) $\Delta_4(\eta^2_{n+9}) = E^{n+1}\pi_3 \circ \eta^2_{n+8}$ *for* $n \geq 1$.
ii) $E^{n+1}\pi_3 \circ \eta^2_{n+8} = 180(\tilde{\nu}'_{n+7} + aiE^{n-4}\sigma')$ *for* $n \geq 4$, *where* $a=0$ *or* 1.

Proof. By $(10)_4$, we have i). By Lemma 6.4 and (21),

$$E^{n+1}\pi_3 \circ \eta^2_{n+8} - 12\tilde{\nu}'_{n+7} \in \operatorname{Ker} p'_* = i'_* \pi_{n+10}(E^{n+1}P^2)$$

for $n \geq 2$. So, by Proposition 11.2 and Lemma 11.3, $E^5\pi_3 \circ \eta^2_{12} \equiv 12\tilde{\nu}'_{11}$ mod $4i\sigma'$. Therefore, $E^5\pi_3 \circ \eta^2_{12} \equiv 180\tilde{\nu}'_{11}$ mod $60i\sigma' = 180i\sigma'$. This completes the proof.

Let $i'' = E^{n+1}i_4 : E^{n+1}P^3 \longrightarrow E^{n+1}P^4$ for $n \geq 0$. Then we have the following

Proposition 11.6. i) $\pi_{14}(E^5P^4) = \{i\sigma', i''\tilde{\nu}'_{11}\} \approx Z_8 + Z_{180}$.
ii) $\pi_{15}(E^6P^4) = \{i\sigma_8, iE\sigma', i''\tilde{\nu}'_{12}\} \approx Z + Z_8 + Z_{180}$.
iii) $\pi_{n+10}(E^{n+1}P^4) = \{i\sigma_{n+3}, i''\tilde{\nu}'_{n+7}\} \approx Z_{16} + Z_{180}$ *for* $n \geq 6$.

Proof. Using $(9)_{4,10}$, $(9)'_{4,10}$, (19), Proposition 10.6 and Lemma 11.5. i), we have a short exact sequence for $n \geq 1$:

$$0 \longrightarrow \pi_{n+11}(S^{n+9}) \xrightarrow{\Delta} \pi_{n+10}(E^{n+1}P^3) \xrightarrow{i''_*} \pi_{n+10}(E^{n+1}P^4) \longrightarrow 0.$$

Hence, by Proposition 11.4 and Lemma 11.5. ii), we have the assertion.

Lemma 11.7. i) $\Delta_5(\iota_{n+11}) = E^{n+1}\pi_4$ *for* $n \geq 0$.
ii) $E^{n+1}\pi_4 = \pm 3(i''\tilde{\nu}'_{n+7} + aiE^{n-4}\sigma') + 15biE^{n-4}\sigma'$ *for* $n \geq 4$, *where* a *is same as in Lemma* 11.5 *and* b *is an odd integer.*

Proof. By $(10)_5$, we have i). By Proposition 6.5. ii) and Lemma 11.3.

i), we have

$$E^{n+1}\pi_4 \equiv \pm 3i''\tilde{\nu}'_{n+7} \bmod iE^{n-4}\sigma'$$

for $n \geq 4$. By Theorem 1.2, $E^{n+1}\pi_4$ is of order 120. Hence, by Lemma 11.5. ii), Proposition 11.6 and its proof, we have ii). This completes the proof.

By $(9)_{5,10}$, Proposition 11.6 and Lemma 11.7, we have the following

Proposition 11.8. i) $\pi_{14}(E^5P^5)=\{i\sigma', i''\tilde{\nu}'_{11}\} \approx \boldsymbol{Z}_{12}$.
ii) $\pi_{15}(E^6P^5)=\{i\sigma_8, iE\sigma', i''\tilde{\nu}'_{12}\} \approx \boldsymbol{Z}+\boldsymbol{Z}_{12}$.
iii) $\pi_{n+10}(E^{n+1}P^5)=\{i\sigma_{n+3}, i''\tilde{\nu}'_{n+7}\} \approx \boldsymbol{Z}_{24}$ *for* $n \geq 6$.

12. Determination of $\pi_{n+11}(E^{n+1}P^m)$ for $n \geq 6$. Hereafter we use the following [15]: $\pi_{n+8}(S^n)=\{\varepsilon_n\} \approx \boldsymbol{Z}_2$ for $3 \leq n \leq 5$, $\pi_{17}(S^9)=\{\sigma_9\eta_{16}, \varepsilon_9, \bar{\nu}_9\} \approx \boldsymbol{Z}_2+\boldsymbol{Z}_2+\boldsymbol{Z}_2$ and $\pi_{n+8}(S^n)=\{\varepsilon_n, \bar{\nu}_n\} \approx \boldsymbol{Z}_2+\boldsymbol{Z}_2$ for $n \geq 10$. $\sigma'\eta_{14}+\varepsilon_7+\bar{\nu}_7 =\eta_7\sigma_8$ and $\varepsilon_n+\bar{\nu}_n=\eta_n\sigma_{n+1}$ for $n \geq 9$. $\bar{\nu}_n=\{\nu_n, \eta_{n+3}, \nu_{n+4}\}$ for $n \geq 7$.

The following secondary compositions contain ε_n: $\{\eta_n, E^{n-2}\nu', 3\nu_{n+4}\}$ for $n \geq 3$; $\{\eta_n, 2\iota_{n+1}, \nu^2_{n+1}\}$ and $\{\eta_n, 2\nu_{n+1}, \nu_{n+4}\}$ for $n \geq 4$; $\{\eta_n, \nu_{n+1}, 2\nu_{n+4}\}$ and $\{2\nu_n, \nu_{n+3}, \eta_{n+6}\}$ for $n \geq 5$.

Lemma 12.1. $2\tilde{\nu}_{n+5}\nu_{n+8}=i\varepsilon_{n+3}$ *for* $n \geq 2$.

Proof. By Proposition 1.4 of [15] and by (16),

$$2\tilde{\nu}_7\nu_{10} \in \{i, \eta_5, \nu_6\}\circ 2\nu_{10}=-i\{\eta_5, \nu_6, 2\nu_9\} \ni i\varepsilon_5$$

mod $(i\eta_5)_*\pi_{13}(S^6)+i_*\pi_{10}(S^5)\circ 2\nu_{10}=0$. So we have the assertion for $n=2$. This completes the proof.

By $(10)_2$ and (11), we have

(22) $$\Delta_2(\sigma_{n+5})=\eta_{n+3}\sigma_{n+4} \text{ for } n \geq 6.$$

Using $(9)_{2,11}$, $(9)'_{2,11}$, (16), (20), (22) and Lemma 12.1, we have the following

Proposition 12.2. i) $\pi_{17}(E^7P^2)=\{i\sigma_9\eta_{16}, \tilde{\nu}_{11}\nu_{14}\} \approx \boldsymbol{Z}_2+\boldsymbol{Z}_4$.
ii) $\pi_{n+11}(E^{n+1}P^2)=\{\tilde{\nu}_{n+5}\nu_{n+8}\} \approx \boldsymbol{Z}_4$ *for* $n \geq 7$.

By $(9)_{3,11}$ and Proposition 12.2, we have the following

Proposition 12.3. i) $\pi_{17}(E^7P^3)=\{i\sigma_9\eta_{16}, i'\tilde{\nu}_{11}\nu_{14}\} \approx \boldsymbol{Z}_2+\boldsymbol{Z}_4$.
ii) $\pi_{n+11}(E^{n+1}P^3)=\{i'\tilde{\nu}_{n+5}\nu_{n+8}\} \approx \boldsymbol{Z}_4$ *for* $n \geq 7$.

Lemma 12.4. i) $E\pi_3\circ\nu_8 \in i'\{i, \eta_3, \nu_4^2\}$ mod $\{i\varepsilon_3, i'\zeta_2\nu_5^2\}$.
ii) $\Delta_3(\nu_{n+9})=E^{n+1}\pi_3\circ\nu_{n+8}=\pm i'\tilde{\nu}_{n+5}\nu_{n+8}$ *for* $n \geq 2$.

Proof. By Proposition 1.4 of [15] and by Lemma 9.3. ii),

$$E\pi_3\circ\nu_8 \in \{i', i\nu', \eta_6\}\circ\nu_8=-i'\{i\nu', \eta_6, \nu_7\}$$

mod $i'\zeta_2\nu_5^2$. By Proposition 1.2 of [15], $\{i\nu', \eta_6, \nu_7\}\subset\{i, \nu'\eta_6, \nu_7\}=\{i, \eta_3\nu_4, \nu_7\}\supset\{i, \eta_3, \nu_4^2\}$ mod $i_*\pi_{11}(S^3)+\pi_8(EP^2)\circ\nu_8$. So, by Proposition 9.2. ii), we have i).

By $(10)_4$, we have the first equality of ii). By Proposition 1.2 of [15], by (16) and Proposition 8.1. iv),

$$\tilde{\nu}_7\nu_{10} \in \{i, \eta_5, \nu_6\}\circ\nu_{10}\subset\{i, \eta_5, \nu_6^2\}$$

mod $i_*\pi_{13}(S^5)+\pi_7(E^3P^2)\circ\nu_7^2=\{i\varepsilon_5, E^2(\zeta_2\nu_5^2)\}$. So, by Proposition 1.3 of [15] and by i),

$$E^3\pi_3\circ\nu_{10}\equiv i'\tilde{\nu}_7\nu_{10} \text{ mod } \{i\varepsilon_5, i'E^2(\zeta_2\nu_5^2)\}.$$

By Propositions 1.3, 1.4 of [15] and by Lemma 9.1,

$$E(\zeta_2\nu_5^2)\in -\{i, \eta_4, 2\iota_5\}\circ\nu_6^2=i\{\eta_4, 2\iota_5, \nu_5^2\}\ni i\varepsilon_4 \text{ mod } 0.$$

So we obtain that $E(\zeta_2\nu_5^2)=i\varepsilon_4$. Therefore, by Lemma 12.1, we have the second equality of ii) for $n=2$. This completes the proof.

Using $(9)_{4,11}$, $(9)'_{4,11}$, Lemmas 11.5, 12.4 and Proposition 12.3, we have the following

Proposition 12.5. i) $\pi_{17}(E^7P^4)=\{i\sigma_9\eta_{16}\}\approx \boldsymbol{Z}_2$.
ii) $\pi_{n+11}(E^{n+1}P^4)\approx 0$ *for* $n\geq 7$.

Lemma 12.6. i) $\Delta_5(\eta_{n+11})=E^{n+1}\pi_4\circ\eta_{n+10}$ *for* $n\geq 0$.
ii) $E^5\pi_4\circ\eta_{14}\equiv 0$ mod $i\sigma'\eta_{14}$.
iii) $E^{n+1}\pi_4\circ\eta_{n+10}=0$ *for* $n\geq 6$.

Proof. By $(10)_5$, we have i). By Lemma 11.7. ii),

$$E^5\pi_4\circ\eta_{14}\equiv i''\tilde{\nu}'_{11}\eta_{14} \text{ mod } i\sigma'\eta_{14}.$$

By Propositions 1.2, 1.4 of [15] and by (21),

$$\begin{aligned}\tilde{\nu}'_{11}\eta_{14} &\in \{i', 2i\nu_7, \nu_{10}\}\circ\eta_{14}\\ &=-i'\{2i\nu_7, \nu_{10}, \eta_{13}\}\\ &\supset -i'i\{2\nu_7, \nu_{10}, \eta_{13}\}\\ &\ni i\varepsilon_7 \text{ mod } i'_*\pi_{14}(E^5P^2)\circ\eta_{14}.\end{aligned}$$

So, by Proposition 11.2. i), $E^5\pi_4\circ\eta_{14}\equiv i\varepsilon_7 \bmod i\sigma'\eta_{14}$. By Lemmas 12.1, 12.4. ii) and by the exact sequence $(9)_{4,11}$ for $n=2$, $i\varepsilon_5=2i''i'\tilde{\nu}_7\nu_{10}=2i''\Delta_4(\nu_{11})=0$. This completes the proof.

Using $(9)_{5,11}$, $(9)'_{5,11}$, Lemmas 11.7. i), 12.6, (3), Theorem 1.2 and Proposition 12.5, we have the following

Proposition 12.7. i) $\pi_{17}(E^7P^5)=\{E^6\zeta_5,\ i\sigma_9\eta_{16}\}\approx \boldsymbol{Z}+\boldsymbol{Z}_2$.
ii) $\pi_{n+11}(E^{n+1}P^5)=\{E^n\zeta_5\}\approx \boldsymbol{Z}$ *for* $n\geq 7$.

13. Proof of Theorem 1.1. In this section we use the following ([1], [4] and [15]): $J_C\pi_3(SU(n))=\pi_{2n+3}(S^{2n})=\{\nu_{2n}\}\approx \boldsymbol{Z}_{24}$ for $n\geq 3$, $J_C\pi_7(SU(n))=2\pi_{2n+7}(S^{2n})=\{2\sigma_{2n}\}\approx\boldsymbol{Z}_{120}$ for $n\geq 5$, $J_C\pi_8(SU(n))\approx 0$ for $n\geq 5$, $J_C\pi_9(SU(n))=\{\eta^2_{2n}\sigma_{2n+2}\}\approx \boldsymbol{Z}_2$ for $n\geq 6$. $J_C\pi_{11}(SU(n))=\pi_{2n+11}(S^{2n})=\{\ddot{\zeta}_{2n}\}\approx \boldsymbol{Z}_{504}$ for $n\geq 7$. $\pi_{n+9}(S^n)=\{\eta^2_n\sigma_{n+2},\ \nu^3_n,\ \mu_n\}\approx \boldsymbol{Z}_2+\boldsymbol{Z}_2+\boldsymbol{Z}_2$ for $n\geq 11$ and $\pi_{n+10}(S^n)=\{\eta_n\mu_{n+1},\ \beta_1(n)\}\approx \boldsymbol{Z}_6$ for $n\geq 12$. $\nu_n\sigma_{n+3}=\sigma_n\nu_{n+7}=0$ and $\{\nu_n,\ 6\nu_{n+3},\ \nu_{n+6}\}=0$ for $n\geq 12$ (cf. (7.13) and (7.14) of [15]). $\beta_1(n)\in\{\alpha_1(n),\ \alpha_1(n+3),\ \alpha_1(n+6)\}$ for $n\geq 5$.

Let c be an integer such that $(c,\ 24)=1$. Then, by Theorem 4.3, we have the following

Proposition 13.1. i) $E^{2(n-2)}g_2=c\nu_{2n}$ *for* $n\geq 3$.
ii) $E^{2(n-3)}g_3\circ E^{2n}\zeta_2=0$ *for* $n\geq 4$.
iii) $\{E^{2(n-4)}g_4\circ E^{2n}\zeta_3\}=\{2\sigma_{2n}\}$ *for* $n\geq 5$.
iv) $E^{2(n-5)}g_5\circ E^{2n}\zeta_4=\eta^2_{2n}\sigma_{2n+2}$ *for* $n\geq 6$.
v) $\{E^{2(n-6)}g_6\circ E^{2n}\zeta_5\}=\{\ddot{\zeta}_{2n}\}$ *for* $n\geq 7$.

Now, we are ready to prove Theorem 1.1.
It is trivial that $B_k\approx 0$ for $k=1,\ 2,\ 4$ or 5.
By (1), Propositions 8.1. ii) and 13.1. i),

$$B_{3,n}=g_{n*}i_*\pi_{2n+3}(E^{2n+1}P^1)=\{E^{2(n-2)}g_2\}=\{\nu_{2n}\}\approx \boldsymbol{Z}_{24} \text{ for } n\geq 3.$$

Using (1), Propositions 8.7, 10.7. iv), 12.7. ii) and 13.1, we have the following: $B_{6,n}=\{\nu^2_{2n}\}\approx\boldsymbol{Z}_2$ for $n\geq 4$, $B_{7,n}=\{2\sigma_{2n}\}\approx\boldsymbol{Z}_{120}$ for $n\geq 5$, $B_{9,n}=\{\eta^2_{2n}\sigma_{2n+2},\ \nu^3_{2n}\}\approx \boldsymbol{Z}_2+\boldsymbol{Z}_2$ for $n\geq 6$ and $B_{11,n}=\{\ddot{\zeta}_{2n}\}\approx \boldsymbol{Z}_{504}$ for $n\geq 7$.
By (1) and Proposition 10.7. iii), $B_{8,n}$ is generated by

$$E^{2(n-5)}g_5\circ i'\tilde{\nu}_{2n+5}=E^{2(n-3)}g_3\circ\tilde{\nu}_{2n+5} \text{ for } n\geq 5.$$

By (1) and Proposition 13.1. i), $E^{2(n-3)}g_3|S^{2n+3}=c\nu_{2n}$ for $n\geq 3$. By Propositions 1.2, 1.7 of [15], by (11) and (16),

$$E^{2(n-3)}g_3\circ\bar{\nu}_{2n+5}\in\{c\nu_{2n},\ \eta_{2n+3},\ \nu_{2n+4}\}=\bar{\nu}_{2n}\ \text{mod}\ 0\ \text{for}\ n\geq 4.$$

Thereofore, $B_{8,n}=\{\bar{\nu}_{2n}\}\approx Z_2$ for $n\geq 5$.

Next we shall prove that $B_{10,n}=\{\beta_1(2n)\}\approx Z_3$ for $n\geq 6$.

By (1) and Prpopsition 11.8. iii), $B_{10,n}$ is generated by

$$E^{2(n-2)}g_2\circ\sigma_{2n+3}\ \text{and}\ E^{2(n-4)}g_4\circ\bar{\nu}'_{2n+7}$$

for $n\geq 6$. By Proposition 13.1. i), $E^{2(n-2)}g_2\circ\sigma_{2n+3}=c\nu_{2n}\sigma_{2n+3}=0$ for $n\geq 6$. Using Propositions 1.2, 1.7 of [15], (1), (13), (21) and Proposition 13.1. i), we see that for $n\geq 4$

$$\begin{aligned}E^{2(n-4)}g_4\circ\bar{\nu}'_{2n+7}&\in\{E^{2(n-3)}g_3,\ \pm 2i\nu_{2n+3},\ \nu_{2n+6}\}\\&\supset\{\pm E^{2(n-3)}g_3\circ i,\ 2\nu_{2n+3},\ \nu_{2n+6}\}\\&\supset\pm c\{\nu_{2n},\ 2\nu_{2n+3},\ \nu_{2n+6}\}\\&=\pm c\{\alpha_1(2n),\ 2\alpha_1(2n+3),\ \alpha_1(2n+6)\}\\&\ni\pm 2c\beta_1(2n)\\&\quad\text{mod}\ E^{2(n-3)}g_{3*}\pi_{2n+10}(E^{2n+1}P^2)+\pi_{2n+7}(S^{2n})\circ\nu_{2n+7}.\end{aligned}$$

By (1), Propositions 11.2. iii) and 13.1, the indeterminacy is equal to $\{\nu_{2n}\sigma_{2n+3}\}+\{\sigma_{2n}\nu_{2n+7}\}=0$ for $n\geq 6$. Therefore we have

$$E^{2(n-4)}g_4\circ\bar{\nu}'_{2n+7}=\pm 2c\beta_1(2n)\ \text{for}\ n\geq 6.$$

Hence, $B_{10,n}=\{\beta_1(2n)\}\approx Z_3$ for $n\geq 6$. Thus the proof of Theorem 1.1 is complete.

Remark. The result of Theorems 1.1 and 4.3 overlaps with the ones of Becker-Schultz [2] and Knapp [6].

Finally, we state the result about generators of the stable homotopy groups of EP^∞. We set

$$\pi_k^S(EP^\infty)=\lim_{n\to\infty}\ \pi_{n+k}(E^{n+1}P^\infty).$$

By summarizing Propositions 8.1, 8.7, 10.7, 11.8 and 12.7, we have the following

Theorem 1.3. *$\pi_k^S(EP^\infty)$ for $k\leq 11$ and generators of them are listed in the following table.*

$k=$	1,2	3	4	5	6	7	8	9	10	11
$\pi_k^S(EP^\infty)\approx$	0	Z	0	Z	Z_2	Z	Z_2	$Z+Z_2$	Z_{24}	Z
gen.		i		$iE^\infty\zeta_2$	$i\nu$	$iE^\infty\zeta_3$	$i'\bar{\nu}$	$iE^\infty\zeta_4,\ i\nu^2$	$i\sigma,\ i''\bar{\nu}'$	$iE^\infty\zeta_5$

Remark. The above result about the group structure of $\pi_k^S(EP^\infty)$ overlaps with the results of Liulevicius [7] and Mosher [10].

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Additional remark, added in proof. Using the notion of the C-projectivity, Hideaki Ōshima obtained the following: $B_{13}=\pi_{13}^S(S^0)$ and $B_{15}=2\pi_{15}^S(S^0)+H$, where $H=0$ or $\{\eta\kappa\} \approx \boldsymbol{Z}_2$.