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NOTES ON STABLE EQUIVARIANT MAPS

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0. Introduction. Stable equivariant maps have been studied by many authors ([1],[7],[8]). In these papers, "the stability theorem" has been established. We say that the stablility theorem holds if the algebraic notion of "infinitesimal stability" is equivalent to the notion of "stability".

In this paper, we will study some properties of stable equivariant maps. In sections 1—4, we study the finite determinacy of stable equivariant map-germs. In section 5, we construct the canonical G-stratification for a stable equivariant maps when both G-manifolds have only a single orbit type. Our main results are Theorems A, B and C. We will formulate A and B in section 1, C in section 5. Since the proof of Theorem A is a direct analogy of the non-equivariant case (cf [4]), we shall omit it. The proof of Theorem B will be given in sections 2—4, in which we will give the notion of G-contact equivalence for equivariant map-germs. All maps should be of class C^∞

1. Formulation of Theorems A and B. Let G be a compact Lie group which acts linearly on $\boldsymbol{R}^n$ and $\boldsymbol{R}^p$. We shall denote by $C_0^G(\boldsymbol{R}^n,\boldsymbol{R}^p)$ the set of all germs at 0 in $\boldsymbol{R}^n$ of smooth G-equivariant maps $\boldsymbol{R}^n \longrightarrow \boldsymbol{R}^p$, and we shall set $C_G^\infty(n,p)=\{f \in C_0^G(\boldsymbol{R}^n,\boldsymbol{R}^p) \mid f(0)=0\}$. If $p=1$ and the action of G on $\boldsymbol{R}$ is trivial, we shall simply write $C_0^G(\boldsymbol{R})$ for $C_0^G(\boldsymbol{R}^n,\boldsymbol{R}^p)$ and $\mathfrak{M}_n^G$ for $C_G^\infty(n,1)$.

Then $C_0^G(\boldsymbol{R}^n)$ is an $\boldsymbol{R}$-algebra in the usual way, and $\mathfrak{M}_n^G$ is its unique maximal ideal. We can make $C_0^G(\boldsymbol{R}^n,\boldsymbol{R}^p)$ a $C_0^G(\boldsymbol{R})$-module in a canonical way. Let $L_G(n)$ be the group of origin preserving equivariant diffeomorphism germs on $\boldsymbol{R}^n$ at 0.

Definition 1.1. Let f and h be in $C_G^\infty(n,p)$. We shall say that f is *G-isomorphic to h* (we write $f \sim_G h$) if there is a pair of maps $(\phi,\psi) \in L_G(n) \times L_G(p)$ such that $\psi \circ f = h \circ \phi$.

If $f \in C_G^\infty(n,p)$, we denote by $j_0^k f$ the k-jet of f at 0.

Definition 1.2. Let $f \in C_G^\infty(n,p)$. We say that f is *G-k-determined relative to* $L_G(n) \times L_G(p)$ if for any $h \in C_G^\infty(n,p)$ such that $j_0^k f = j_0^k h$ we have $f \sim_G h$. We say that f is *G-finitely determined relative to* $L_G(n) \times L_G(p)$ if f is G-k-determined relative to $L_G(n) \times L_G(p)$ for some integer k.

In this paper, we always have a natural G-action on TR^p which is induced by the action on R^p. For $f \in C_G^\infty(n,p)$, let $\theta_G(f)$ be the set of smooth G-equivariant section germs of $f^*TR^p \longrightarrow R^n$ at 0. Then, it has the natural $C_0^G(R^n)$-module structure. Let $\theta_G(n) = \theta_G(1_{R^n})$. Define maps

$$tf : \theta_G(n) \longrightarrow \theta_G(f) \text{ by } tf(\xi) = df \circ \xi,$$
$$\omega f : \theta_G(p) \longrightarrow \theta_G(f) \text{ by } \omega f(\eta) = \eta \circ f.$$

Then tf is a homomorphism of $C_0^G(R^n)$-modules and ωf is a homomorphism of $f^*(C_0^G(R^p))$-modules. We now define

$$d(f, G) = \dim_R(\theta_G(f)/tf(\theta_G(n)) + \omega f(\theta_G(p))).$$

Then we have the following theorem.

Theorem A. *Let* $f \in C_G^\infty(n,p)$. *Then* f *is G-finitely determined relative to* $L_G(n) \times L_G(p)$ *if and only if* $d(f,G) < +\infty$.

In order to formulate Theorem B, we need the following propositions.

Proposition 1.3 (Hilbert's finitude theorem). *We denote by* $R_G[X_1, \cdots, X_n]$ *the ring of G-invariant polynomials. Then there exist finitely many homogeneous G-invariant polynomials* $\rho_1, \cdots, \rho_m$ *which generate* $R_G[X_1, \cdots, X_n]$ *as an R-algebra. (These polynomials will be called a Hilbert homogeneous basis over* R^n*).*

Proposition 1.4. *There exists a finite set of polynomial maps* $P_1, \cdots, P_h$ *which generates* $C_0^G(R^n, R^p)$ *(resp.* $C_0^G(R^n \times R, R^p)$*) over* $C_0^G(R^n)$ *(resp.* $C_0^G(R^n \times R)$*), where G acts trivially on* R^n.

The proof of Proposition 1.4 is straightforward, using a parametrized version of Schwartz's finitude theorem in [9]. Throughout the remainder of this paper, we fix a generating set of homogeneous polynomial maps, $\{\rho_1, \cdots, \rho_m\}$ and $\{P_1, \cdots, P_h\}$ in the above propositions. We now define the following:

a) $\text{degree}(P_i) = \min\{\text{degree}(P_i^j) \mid 1 \leq j \leq p\}$ for $i = 1, \cdots, h$ (where $P_i = (P_i^1, \cdots, P_i^p)$ is the coordinate representation).

b) $r = \max\{\text{degree}(P_i) \mid 1 \leq i \leq h\}$,

c) $s = \dim_R(\theta_G(p)/\mathfrak{M}_p^G \theta_G(p))$,

d) $D = \max\{\text{degree}(\rho_i) \mid 1 \leq i \leq m\}$.

Definition 1.5. Let $f \in C_G^\infty(n,p)$. We say that f is *infinitesimally stable* if $\theta_G(f)=tf(\theta_G(n))+\omega f(\theta_G(p))$.

Remark. In [8], F. Ronga has defined the notion of a stable equivariant map germ and shown that it is equivalent to the notion of infinitesimally stable map germ. Then, we will take "stable" as a shorthand expression for infinitesimally stable.

Then we have the following theorem.

Theorem B. *Let $f \in C_G^\infty(n,p)$ be stable. Then f is G-$\{(s+1)D+r\}$-determined relative to $L_G(n)\times L_G(p)$.*

Remark. This theorem reduces to Proposition (3.6) in [5], when $G=1$.

2. ***G*-contact equivalence.** In this section, we introduce some important tools in order to prove Theorem B. Let $\sigma_1,\cdots,\sigma_l$ be a Hilbert homogeneous basis over $\boldsymbol{R}^p$ and σ be the map defined by $\sigma(x)=(\sigma_1(x),\cdots,\sigma_l(x))$ for $x \in \boldsymbol{R}^p$.

Definition 2.1. a) A triple $(\phi\ \Phi\ \psi)$ consisting of germs of equivariant diffeomorphisms $\phi:(\boldsymbol{R}^n,0)\longrightarrow(\boldsymbol{R}^n,0)$, $\Phi:(\boldsymbol{R}^n\times\boldsymbol{R}^p,0)\longrightarrow(\boldsymbol{R}^n\times\boldsymbol{R}^p,0)$ and $\psi:(\boldsymbol{R}^n\times\boldsymbol{R}^l,0)\longrightarrow(\boldsymbol{R}^n\times\boldsymbol{R}^l,0)$ (where G acts on $\boldsymbol{R}^l$ trivially) such that the following diagrams commute:

$$\begin{array}{ccccc}
(\boldsymbol{R}^n,0) & \xrightarrow{\ \iota\ } & (\boldsymbol{R}^n\times\boldsymbol{R}^p,0) & \xrightarrow{\ \pi\ } & (\boldsymbol{R}^n,0) \\
\phi\downarrow & & \downarrow\Phi & & \phi\downarrow \\
(\boldsymbol{R}^n,0) & \xrightarrow{\ \iota\ } & (\boldsymbol{R}^n\times\boldsymbol{R}^p,0) & \xrightarrow{\ \pi\ } & (\boldsymbol{R}^n,0)
\end{array}$$

and

$$\begin{array}{ccccc}
(\boldsymbol{R}^n,0) & \xrightarrow{\ \iota\ } & (\boldsymbol{R}^n\times\boldsymbol{R}^l,0) & \xrightarrow{\ \pi\ } & (\boldsymbol{R}^n,0) \\
\phi\downarrow & & \downarrow\psi & & \phi\downarrow \\
(\boldsymbol{R}^n,0) & \xrightarrow{\ \iota\ } & (\boldsymbol{R}^n\times\boldsymbol{R}^l,0) & \xrightarrow{\ \pi\ } & (\boldsymbol{R}^n,0)
\end{array}$$

(where ι is the canonical inclusion and π is the canonical projection) is called a *G-contact equivalence with respect to σ* (or *$\mathcal{K}_G^\sigma$-equivalence*). If $\phi=1$, we call it a *$\mathcal{C}_G^\sigma$-equivalence.*

b) Equivariant map germs f and $h:(\boldsymbol{R}^n,0)\longrightarrow(\boldsymbol{R}^p,0)$ are *G-contact equivalent with respect to σ* (resp. *$\mathcal{K}_G^\sigma$-equivalent, $\mathcal{C}_G^\sigma$-equivalent*) if there exists a G-contact equivalence with respect to σ (resp. $\mathcal{K}_G^\sigma$-equivalence,

$\mathcal{C}\mathcal{E}$-equivalence) $(\phi\ \Phi\ \psi)$ such that $(1,f)\circ\phi=\Phi\circ(1,h)$ and $(1,\sigma\circ f)\circ\phi=\psi\circ(1,\sigma\circ h)$ where $(1,f)\ (x)=(x,f(x))$.

If $f\in C_G^\infty(n,p)$, we let $I_G(f)=f^*(\mathfrak{M}_p^G)C_0^G(\boldsymbol{R}^n)$. By Schwartz's finitude theorem in [9], $I_G(f)$ is the ideal in $C_0^G(\boldsymbol{R}^n)$ generated by $\sigma_1\circ f,\cdots,\sigma_l\circ f$. We also introduce the following ideal in $C_0^G(\boldsymbol{R}^n\times\boldsymbol{R}^l)$:

$$Z_G(f)=\{u\mid\ u\,|\,\mathrm{graph}(\sigma\circ f)=0\}.$$

Let $\iota:(\boldsymbol{R}^n,0)\longrightarrow(\boldsymbol{R}^n\times\boldsymbol{R}^l,0)$ be the canonical inclusion.

Lemma 2.2. $I_G(f)=\iota^*(Z_G(f))$.

Proof. Let $(z_1,\cdots,z_l)$ be the canonical coordinates on $\boldsymbol{R}^l$. Then $Z_G(f)$ is generated by $\pi_2^*(z_i)-\pi_1^*(\sigma_i\circ f)(i=1,\cdots,l)$, where $\pi_1:(\boldsymbol{R}^n\times\boldsymbol{R}^l,0)\longrightarrow(\boldsymbol{R}^n,0)$ and $\pi_2:(\boldsymbol{R}^n\times\boldsymbol{R}^l,0)\longrightarrow(\boldsymbol{R}^l,0)$ are canonical projections. Hence, $\iota^*(Z_G(f))$ is the ideal generated by $\sigma_1\circ f,\cdots,\sigma_l\circ f$.

Proposition 2.3. *If f, $h\in C_G^\infty(n,p)$ are $\mathcal{C}\mathcal{E}$-equivalent, then $I_G(f)=I_G(h)$.*

Proof. By the definition, there exists an invertible equivariant germ $\psi:(\boldsymbol{R}^n\times\boldsymbol{R}^l,0)\longrightarrow(\boldsymbol{R}^n\times\boldsymbol{R}^l,0)$ such that $\psi(\boldsymbol{R}^n\times0,0)=1$ and $\psi(\mathrm{graph}(\sigma\circ f))=\mathrm{graph}(\sigma\circ h)$. We have $\psi^*(Z_G(h))=Z_G(f)$ and $\iota^*\circ\psi^*=\iota^*$. Hence, $I_G(h)=\iota^*(Z_G(h))=\iota^*\circ\psi^*(Z_G(f))=\iota^*Z_G(f)=I_G(f)$.

By the definition and Proposition 2.3, we have the following theorem.

Theorem 2.4. *If f, $h\in C_G^\infty(n,p)$ are $\mathcal{K}\mathcal{E}$-equivalent, then there is an invertible equivariant map germ $\phi:(\boldsymbol{R}^n,0)\longrightarrow(\boldsymbol{R}^n,0)$ such that $\phi^*(I_G(f))=I_G(h)$.*

Remark. We can define "*G-k-determined relative to $\mathcal{K}\mathcal{E}$*" in the same way as in the case of $L_G(n)\times L_G(p)$.

3. The tangent space of an orbit of $L_G(n)\times L_G(p)$. If k is a non-negative integer we denote by $J^k(n,p)$ the space of all k-jets at 0 of germs $(\boldsymbol{R}^n,0)\longrightarrow(\boldsymbol{R}^p,0)$. We see that $J^k(n,p)$ is a finite dimensional real vector space. The set of k-jets at 0 of equivariant map germs will be denoted by $J_G^k(n,p)$. It is clear that $J_G^k(n,p)$ is a linear subspace of $J^k(n,p)$.

Definition 3.1. $L_G^k(n)$ is the Lie group consisting of k-jets of elements of $L_G(n)$.

We have an analytic action of $L^k_G(n)\times L^k_G(p)$ on $J^k_G(n,p)$: $(j^k_0\phi, j^k_0\psi)\cdot j^k_0 f = j^k_0(\psi\circ f\circ\phi^{-1})$. We now define an $\boldsymbol{R}$-linear mapping

$$\pi^k : \tilde{\theta}_G(f) \longrightarrow T_z J^k_G(n,p)=J^k_G(n,p)$$

by $\pi^k(\xi)=j^k_0\xi$, whose kernel is $\mathfrak{M}^{k+1}_n\theta(f)\cap\theta_G(f)$. Here we set $\tilde{\theta}_G(f)=\{\xi\in\theta_G(f)\mid\xi(0)=0\}$ and $z=j^k_0 f$. When $G=1$, $\theta_G(f)$ and $\mathfrak{M}^G_n$ are denoted by $\theta(f)$ and $\mathfrak{M}_n$ respectively.

Proposition 3.2. *Let $f\in C^\infty_G(n,p)$ and $j^k_0 f=z$. Then*

$$T_z(L^k_G(n)\times L^k_G(p)(z))=\pi^k(tf(\tilde{\theta}_G(n))+\omega f(\tilde{\theta}_G(p))).$$

The proof is parallel to the non-equivariant case ([4]).

The projection from $J^r_G(n,p)$ to $J^k_G(n,p)$ is denoted by $\pi^r_{G\ k}$ for any integer $r\geq k$. For $z=j^k_0 f$, we set $E_{G,z}=(\pi^r_{G\ k})^{-1}(z)$. Then, we have $T_z(E_{G,z})=\pi^r(\mathfrak{M}^{k+1}_n\theta(f)\cap\theta_G(f))$. Hence, we have the following theorem.

Proposition 3.3. *Let $f\in C^\infty_G(n,p)$ be G-r-determined relative to $L_G(n)\times L_G(p)$. For any integer $k\leq r$, the following conditions are equivalent:*

1) *f is G-k-determined relative to $L_G(n)\times L_G(p)$.*
2) *For any $h\in C^\infty_G(n,p)$ such that $j^k_0 f=j^k_0 h$, we have*

$$th(\tilde{\theta}_G(n))+\omega h(\tilde{\theta}_G(p))+\mathfrak{M}^{r+1}_n\theta(h)\cap\theta_G(h)\supset\mathfrak{M}^{k+1}_n\theta(h)\cap\theta_G(h).$$

Proof. Since $E_{G,z}$(where $z=j^k_0 h$) is an affine subspace of $J^k_G(n,p)$, we can apply Proposition 3.2 and Lemma (3.1) in [5].

4. The proof of Theorem B. By Theorem A and the definition of stability, the stable equivariant map germ is G-finitely determined relative to $L_G(n)\times L_G(p)$. By Proposition 3.3, it is enough to show the following theorem in order to prove Theorem B.

Theorem 4.1. *Let $f\in C^\infty_G(n,p)$ be stable. For any $h\in C^\infty_G(n,p)$ such that $j^{(s+1)D+r}_0 h=j^{(s+1)D+r}_0 f$, we have*

$$th(\tilde{\theta}_G(n))+\omega h(\tilde{\theta}_G(p))\supset\mathfrak{M}^{(s+1)D+r+1}_n\theta(h)\cap\theta_G(h).$$

For the proof of Theorem 4.1, we need the following algebraic tool and some lemmas. For $f\in C^\infty_G(n,p)$, we define an $\boldsymbol{R}$-homomorphism

$$S_{f,G}:\theta_G(p)/\mathfrak{M}^G_p\theta_G(p)\longrightarrow\theta_G(f)/(tf(\theta_G(n))+f^*(\mathfrak{M}^G_p)\theta_G(f))$$

by $S_{f,G}(\{\xi\})=[\omega f(\xi)]$.

Lemma 4.2. *f is stable if and only if $S_{f,G}$ is onto.*

Proof. "Only if" is clear. If $S_{f,G}$ is surjective, then $\theta_G(f)=tf(\theta_G(n))+\omega f(\theta_G(p))+f^*(\mathfrak{M}_p^G)\theta_G(f)$. Then, by the equivariant preparation theorem (cf. [4]), we have $\theta_G(f)=tf(\theta_G(n))+\omega f(\theta_G(p))$.

Next, we need some results about $\mathcal{K}\mathcal{E}$-equivalences.

Lemma 4.3. *Let $(\phi\ \Phi\ \psi)$ be a $\mathcal{K}\mathcal{E}$-equivalence between equivariant map germs f and h and $c_{(\phi\ \Phi)}: \theta_G(f) \longrightarrow \theta_G(h)$ be the $C_0^G(\boldsymbol{R})$-module isomorphism defined by $c_{(\phi\ \Phi)}(\xi)=d\Phi_1\circ(1\ \xi)\circ\phi^{-1}$(where $\Phi(x,y)=(\phi(x), \Phi_1(x,y))$). Then $c_{(\phi\ \Phi)}$ induces an $\boldsymbol{R}$-vector space isomorphism*

$$\theta_G(f)/(tf(\theta_G(n))+f^*(\mathfrak{M}_p^G)\theta_G(f)) \cong \theta_G(h)/(th(\theta_G(n))+h^*(\mathfrak{M}_p^G)\theta_G(h)).$$

Proof. We may write $c_{(\phi\ \Phi)}=c_{(\phi\ \phi\times 1)}\circ c_{(1\ \Phi')}$, where $\Phi'=(\Phi^{-1}\times 1)\circ\Phi$, and so it is sufficient to show that $c_{(\phi\ \phi\times 1)}$ and $c_{(1\ \Phi')}$ are well defined.

1) $c_{(\phi\ \phi\times 1)}$; we have $h=f\circ\phi^{-1}$, and $c_{(\phi\ \phi\times 1)}=\omega\phi^{-1}$. Then $c_{(\phi\ \phi\times 1)}tf(\theta_G(n))\subset th(\theta_G(n))$ and $c_{(\phi\ \phi\times 1)}(f^*(\mathfrak{M}_p^G)\theta_G(f))\subset h^*(\mathfrak{M}_p^G)\theta_G(h)$. Thus, $c_{(\phi\ \phi\times 1)}$ is well defined.

2) $c_{(1\ \Phi')}$; we have $c_{(1\ \Phi)}=t\Phi_1'|\theta_G(f)$, where $t\Phi_1': \theta_G(1,f) \longrightarrow \theta_G(h)$ is defined by $t\Phi_1'(\xi)=d\Phi_1'\circ\xi$. By Proposition 2.3, $I_G(f)=I_G(h)$, so that $c_{(1\ \Phi')}(f^*(\mathfrak{M}_p^G)\theta_G(f))=h^*(\mathfrak{M}_p^G)\theta_G(h)$. Also, we have $c_{(1\ \Phi')}(tf(\theta_G(n))\subset th(\theta_G(n))+h^*(\mathfrak{M}_p^G)\theta_G(h)$. So $c_{(1\ \Phi')}$ is well defined.

We have the following theorem about G-finitely determined map germs relative to $\mathcal{K}\mathcal{E}$.

Theorem 4.4. *Let $f\in C_G^\infty(n,p)$ be such that*

$$(*)\qquad (\mathfrak{M}_p^G)^k\theta_G(f)\subset tf(\theta_G(n))+f^*(\mathfrak{M}_p^G)\theta_G(f)$$

for some integer k. Then f is G-$\{(k+1)D+r\}$-determined relative to $\mathcal{K}\mathcal{E}$.

The structure of the proof is an equivariant extension of Mather's method (cf. Mather [4], Section 5). We will give an outline of the proof in the last of this section.

Next, we have an estimate of the order of $\mathcal{K}\mathcal{E}$-determinacy of stable map germs.

Lemma 4.5. *Let $f\in C_G^\infty(n,p)$ be stable. Then we have $(\mathfrak{M}_n^G)^s\theta_G(f)\subset tf(\theta_G(n))+f^*(\mathfrak{M}_p^G)\theta_G(f)$. Hence, f is G-$\{(s+1)D+r\}$-determined relative* *to* $\mathcal{K}_G^{\mathcal{E}}$.

Proof. Since f is stable, $tf(\theta_G(n))+\omega f(\theta_G(p))$ has a $C_0^G(\boldsymbol{R}^n)$-module structure. Let

$$V=(tf(\theta_G(n))+\omega f(\theta_G(p))+f^*(\mathfrak{M}_p^G)\theta_G(f)/(tf(\theta_G(n))+f^*(\mathfrak{M}_p^G)\theta_G(f)+(\mathfrak{M}_n^G)^{s+1}\theta_G(f)).$$

Then

$$\dim_R V \leq \dim_R(\omega f(\theta_G(p))/\omega f(\mathfrak{M}_p^G\theta_G(p)) \leq \dim_R(\theta_G(p)/\mathfrak{M}_p^G\theta_G(p))=s.$$

By Nakayama's lemma, we have the required result.

We must study relations between $C_0^G(\boldsymbol{R}^n,\boldsymbol{R}^p)$ and $C_0^\infty(\boldsymbol{R}^n,\boldsymbol{R}^p)$. Here, $C_0^\infty(\boldsymbol{R}^n,\boldsymbol{R}^p)$ is the set of all germs at 0 of smooth mappings $\boldsymbol{R}^n \longrightarrow \boldsymbol{R}^p$. This set is also a module over the local algebra $C_0^\infty(\boldsymbol{R}^n)$. Let $\rho_1,\cdots,\rho_m$ be a Hilbert homogeneous basis over $\boldsymbol{R}^n$. Set $d=\min\{\text{degree}(\rho_i)|i=1,\cdots,m\}$.

Lemma 4.6 (Ronga [8]). i) $\mathfrak{M}_n^G \subset \mathfrak{M}_n^d$.
ii) $\mathfrak{M}_n^{qD} \cap C_0^G(\boldsymbol{R}^n) \subset (\mathfrak{M}_n^G)^q$ *for any positive integer* q.

Corollary 4.7. i) $\mathfrak{M}_n^G C_0^G(\boldsymbol{R}^n,\boldsymbol{R}^p) \subset \mathfrak{M}_n^d C_0^\infty(\boldsymbol{R}^n,\boldsymbol{R}^p) \cap C_0^G(\boldsymbol{R}^n,\boldsymbol{R}^p)$.
ii) $\mathfrak{M}_n^{qD+r} C_0^\infty(\boldsymbol{R}^n,\boldsymbol{R}^p) \cap C_0^G(\boldsymbol{R}^n,\boldsymbol{R}^p) \subset (\mathfrak{M}_n^G)^q C_0^G(\boldsymbol{R}^n,\boldsymbol{R}^p)$ *for any positive integer* q.

Proof. i) Trivial.
ii) If $f\in M_n^{qD+r}C_0^\infty(\boldsymbol{R}^n,\boldsymbol{R}^p) \cap C_0^G(\boldsymbol{R}^n,\boldsymbol{R}^p)$, then f may be written as $f(x)=\sum_{i=1}^h H_i(x)P_i(x)$, where $H_i(x) \in C_0^G(\boldsymbol{R}^n) \cap \mathfrak{M}_n^{h_i}$ for some integer h_i. Let $d_i=\text{degree}(P_i(x))$. Since D is the maximum of d_i, then $h_i+d_i \geq qD+r$. Hence, $f(x) \in (\mathfrak{M}_n^G \cap \mathfrak{M}_n^{qD})C_0^G(\boldsymbol{R}^n,\boldsymbol{R}^p)$. By Lemma 4.6, the proof is completed.

Then, we have tools in order to prove Theorem 4.1.

Proof of Theorem 4.1. Let $h\in C_G^\infty(n,p)$ be such that $j_0^{(s+1)D+r}h=j_0^{(s+1)D+r}f$. Then f and h are $\mathcal{K}_G^{\mathcal{E}}$-equivalent, because f is G-$\{(s+1)D+r\}$-determined relative to $\mathcal{K}_G^{\mathcal{E}}$. If $\xi\in\theta_G(n)$, then $tf(\xi)-th(\xi)\in\mathfrak{M}_n^{(s+1)D+r}\theta(f) \cap \theta_G(f)$. If $v\in C_0^G(\boldsymbol{R}^p)$, then $f^*(v)-h^*(v)\in\mathfrak{M}_n^{(s+1)D+r+1}C_0^\infty(\boldsymbol{R}^n)\cap C_0^G(\boldsymbol{R}^n)$. Hence, by Lemma 4.5 and Corollary 4.7, $th(\theta_G(n))+h^*(\mathfrak{M}_p^G)\theta_G(h) \subset tf(\theta_G(n))+f^*(\mathfrak{M}_p^G)\theta_G(f)$. Here, we identify $\theta_G(f)$ with $C_0^G(\boldsymbol{R}^n,\boldsymbol{R}^p)$. Since f and h are $\mathcal{K}_G^{\mathcal{E}}$-equivalent, by Lemma 4.3,

$$tf(\theta_G(n))+f^*(\mathfrak{M}_p^G)\theta_G(f)=th(\theta_G(n))+h^*(\mathfrak{M}_p^G)\theta_G(h).$$

So $S_{f,G}$ and $S_{h,G}$ can be identified. Because f is stable, by Lemma 4.2, $S_{f,G}=S_{h,G}$ are surjective, and hence h is stable.

By Lemma 4.5, $(\mathfrak{M}_n^G)^s\theta_G(h)\subset th(\theta_G(n))+h^*(\mathfrak{M}_p^G)\theta_G(h)$, so that $(\mathfrak{M}_n^G)^{s+1}\theta_G(h)\subset th(\mathfrak{M}_n^G\theta_G(n))+h^*(\mathfrak{M}_p^G)\mathfrak{M}_n^G(th(\theta_G(n))+\omega h(\theta_G(p))\subset th(\mathfrak{M}_n^G\theta_G(n))+\omega h(\mathfrak{M}_p^G\theta_G(p))$. Therefore, by Corollary 4.7, $\mathfrak{M}_n^{(s+1)D+r}\theta_G(h)\cap\theta_G(h)\subset th(\mathfrak{M}_n^G\theta_G(n))+\omega h(\mathfrak{M}_p^G\theta_G(p))\subset th(\tilde{\theta}_G(n))+\omega h(\tilde{\theta}_G(p))$. This completes the proof.

Outline of the proof of Theorem 4.4. Let $f, h\in C_G^\infty(n,p)$. We let $H:(\boldsymbol{R}^n\times\boldsymbol{R},0\times\boldsymbol{R})\longrightarrow(\boldsymbol{R}^n\times\boldsymbol{R},0\times\boldsymbol{R})$ be given by $H(x,t)=((1-t)f(x)+tf(x),t)$. For each $a\in\boldsymbol{R}$, we define $H^a=H|(\boldsymbol{R}^n\times\boldsymbol{R},(0,a))$ and $H_a=H^a|(\boldsymbol{R}^n\times a,(0,a))$. Let π_n be the germ of the canonical projection $\boldsymbol{R}^n\times\boldsymbol{R}\longrightarrow\boldsymbol{R}^n$. By Prop. 1.4, $(\mathfrak{M}_n^G)^k\theta_G(f)$ generates $(\mathfrak{M}_n^G)^k\theta_G(\pi_p\circ H^a)$ as $C_0^G(\boldsymbol{R}^n\times\boldsymbol{R})$-module. By (*), if $j_0^{(k+1)D+r}f=j_0^{(k+1)D+r}h$, then $(\mathfrak{M}_n^G)^k\theta_G(\pi_p\circ H^a)\subset(H^a)^*(\mathfrak{M}_p^G)\mathfrak{M}_n^G\theta_G(\pi_p\circ H^a)+t_1H^a(\mathfrak{M}_n^G\theta_G(\pi_n))+(\mathfrak{M}_n^G)^{k+1}\theta_G(\pi_p\circ H^a)$. Here, we set $t_1H^a(\xi)=dH_t^a\circ\xi$. By Nakayama's lemma and multiplying both sides by $\mathfrak{M}_n^G$, we obtain

$$(\mathfrak{M}_n^G)^{k+1}\theta_G(\pi^p\circ H^a)\subset(H^a)^*(\mathfrak{M}_p^G)\mathfrak{M}_n^G\theta_G(\pi_p\circ H^a)+t_1H^a(\mathfrak{M}_n^G\theta_G(\pi_n)).$$

We now have

$$\frac{\partial H^a}{\partial t}\in\mathfrak{M}_n^{(k+1)D+r+1}\theta_G(\pi_p\circ H^a)\cap\theta_G(\pi_p\circ H^a)\subset(\mathfrak{M}_n^G)^{k+1}\theta_G(\pi_p\circ H^a).$$

It follows that there are $\eta\in(H^a)^*(\mathfrak{M}_p^G)\mathfrak{M}_n^G\theta_G(\pi_p\circ H^a)$ and $\xi\in\mathfrak{M}_n^G\theta_G(\pi_n)$ such that $\frac{\partial H^a}{\partial t}=\eta+t_1H^a(\xi)$. Since $(H^a)^*(\mathfrak{M}_p^G)\,\mathfrak{M}_n^G\theta_G(\pi_p\circ H^a)\subset(H^a)^*(\mathfrak{M}_p)\,\mathfrak{M}_n\theta(\pi_p\circ H^a)$, there exist $u_{ij}\in\mathfrak{M}_nC_0^\infty(\boldsymbol{R}^n\times\boldsymbol{R})$ such that $\eta=\sum_{i,j=1}^p(H^a)^*(y_j)u_{ij}\left(\frac{\partial}{\partial y_j}\circ\pi_p\circ H^a\right)$, where $(y_1,\cdots,y_p)$ denotes the canonical coordinates for $\boldsymbol{R}^p$. We now define a $C_0^G(\boldsymbol{R}^n\times\boldsymbol{R})$-homomorphism

$$t\sigma:\theta_G(\pi_p\circ H^a)\longrightarrow\theta_G(\sigma\circ\pi_p\circ H^a)\text{ by }t\sigma(\zeta)=d\sigma\circ\zeta.$$

Then we have

$$\frac{\partial\sigma\circ H^a}{\partial t}=t\sigma\left(\frac{\partial H^a}{\partial t}\right)=t\sigma(\eta)+t_1(\sigma\circ H^a)(\xi).$$

Since $(H^a)^*(\mathfrak{M}_p^G)$ is generated by $\sigma_1\circ H^a,\cdots,\sigma_l\circ H^a$, then there are $w_{ij}\in\mathfrak{M}_n^GC_0^G(\boldsymbol{R}^n\times\boldsymbol{R})$ such that

$$t\sigma(\eta)=\sum_{i,j=1}^l(\sigma_j\circ H^a)w_{ij}\left(\frac{\partial}{\partial z_j}\circ\sigma\circ\pi_p\circ H^a\right),$$

where $(z_1,\cdots,z_l)$ denotes the canonical coordinates for $\boldsymbol{R}^l$. Hence, we have the following formulas:

$$\frac{\partial H^a}{\partial t}=\sum_{i,j=1}^{p}(H^a)^*(y_j)u_{ij}\Big(\frac{\partial}{\partial y_i}\circ\pi_p\circ H^a\Big)+t_1H^a(\xi),$$
$$\frac{\partial\sigma\circ H^a}{\partial t}=\sum_{i,j=1}^{l}(\sigma_j\circ H^a)w_{ij}\Big(\frac{\partial}{\partial z_i}\circ\sigma\circ\pi_p\circ H^a\Big)+t_1(\sigma\circ H^a)(\xi).$$

Let $\pi_{n+p}:(\boldsymbol{R}^n\times\boldsymbol{R}^p\times\boldsymbol{R},(0,0,a))\longrightarrow(\boldsymbol{R}^n\times\boldsymbol{R}^p,(0,0))$, $\pi_{n+l}:(\boldsymbol{R}^n\times\boldsymbol{R}^l\times\boldsymbol{R},(0,0,a))\longrightarrow(\boldsymbol{R}^n\times\boldsymbol{R}^l,(0,0))$ denote projections. Let $\lambda\in\theta(\pi_{n+p})$ and $\mu\in\theta(\pi_{n+l})$ be given by

$$\lambda=\sum_{i,j=1}^{p}y_ju_{ij}\frac{\partial}{\partial y_i}-\xi \quad\text{and}\quad \mu=\sum_{i,j=1}^{l}z_jw_{ij}\frac{\partial}{\partial z_i}-\xi.$$

Let $\tilde{\Phi}$ be the integral of λ (resp. $\tilde{\Psi}$ be the integral of μ) and $\tilde{\phi}$ be the integral of ξ. Using the standard method of the non-equivariant case ([5], Section 5), there are local diffeomorphisms Φ, Ψ and ϕ such that the diagrams in Definition 2.1 are commutative and satisfy $\Phi\circ(1,f)\circ\phi^{-1}=(1,h)$ and $\Psi\circ(1,\sigma\circ f)\circ\phi^{-1}=(1,\sigma\circ h)$. But, in this case, Φ may not be equivariant. We can avoid this situation by integrating over the group G. (i.e. $\tilde{\Phi}(x,y)=(\phi(x),\int_G g^{-1}\Phi_1(gx,gy)dg)$, where $\Phi(x,y)=(\phi(x),\Phi_1(x,y))$).

5. Canonical stratification for stable equivariant maps. The nice range for stable equivariant maps may be variable with respect to group actions. Hence, there is a natural reason that we consider topologically stable equivariant maps, whose definition is the direct analogy of the non-equivariant case (cf. [6]).

Problem. *Is the set of topologically stable equivariant maps open and dense in $C_G^\infty(X,Y)$?*

In relation to this problem, we shall show that every C^∞-stable equivariant map have an "equivariant canonical stratification" when X and Y have only one orbit type respectively. In this situation, the definition of G-transversality is the same as in the non-equivariant case.

Definition 5.1. Let $f:X\longrightarrow Y$ be a smooth equivariant mapping. We call $(\mathcal{X},\mathcal{Y})$ a *G-Thom stratification* of f if the following conditions hold:

(1) $\mathcal{X}$, $\mathcal{Y}$ are Whitney stratifications of X, Y respectively whose strata are G-submanifolds.

(2) For any stratum $U\in\mathcal{X}$, there exists a stratum $V\in\mathcal{Y}$ such that $f(U)\subset V$ and $f|U:U\longrightarrow V$ is a submersion.

(3) For any strata U, $V\in\mathcal{X}$, these satisfy the condition (a_f) (see Mather [6]).

Now, the definition of the orbit type at x is the conjugacy class (G_x) of the isotropy subgroup at $x \in X$. If X has only one orbit type (H), by the differentiable slice theorem, the invariant tubular neighbourhood about any orbit is G-isomorphic to $G/H \times V$. Here, V denotes an open neighbourhood of the origin in the normal space of the orbit at the point and H acts trivially on V. In this case, the canonical projection $X \longrightarrow X/G$ has a fibre bundle structure whose local triviality is given by $G/H \times V \longrightarrow V$. For $f \in C^\infty_G(X, Y)$, we define a smooth mapping

$$\bar{f} : X/G \longrightarrow Y/G$$

by $\bar{f}(\{x\})=[f(x)]$.

Lemma 5.2. *Let X, Y be G-manifolds which have only one orbit type (H), (H') respectively. The mapping $\Pi : C^\infty_G(X, Y) \longrightarrow C^\infty(X/G, Y/G)$ defined by $\Pi(f)=\bar{f}$ is the open mapping.*

The proof is given by the fact that $J^r_G(X, Y) \longrightarrow J^r(X/G, Y/G)$ is the fibre bundle in this case. Here, $J^r_G(X, Y)$ denotes the set of jets which are represented by local equivariant mappings.

Corollary 5.3. *Let X, Y be same as in the above lemma. If $f \in C^\infty_G(X, Y)$ is a C^∞-stable equivariant mapping, then $\bar{f} : X/G \longrightarrow Y/G$ is a C^∞-stable mapping.*

Corollary 5.4. *Let Y, Y be same as in the above lemma. If $f \in C^\infty_G(X, Y)$ is a C^∞-stable equivariant mapping, then there is a Thom stratification $(\bar{\mathcal{X}}, \bar{\mathcal{Y}})$ of $\bar{f}$.*

Proof. By Corollary 5.3, $\bar{f}$ is a C^∞-stable mapping. Hence, by a theorem of Mather ([6]), $\bar{f}$ has a Thom stratification.

We now have the following theorem.

Theorem C. *Let X, Y be same as in Lemma* 5.2. *If $f \in C^\infty_G(X, Y)$ is a C^∞-stable equivariant mapping, then there is a G-Thom stratification $(\mathcal{X}, \mathcal{Y})$ of f.*

Proof. We have open coverings $\{G/H \times U\}$ and $\{G/H' \times V\}$ of X and Y respectively which have the following properties:

(1) $G/H \times U$ and $G/H' \times V$ are tubular neighbourhoods of orbits.

(2) For any $G/H \times U$, there is a $G/H' \times V$ such that $f(G/H \times U) \subset G/H' \times V$.

If $G/H\times U$ and $G/H'\times V$ have the property (2), then $\bar{f}(U)\subset V$. We now define

$$\mathcal{X}_U=\{G/H\times(S\cap U)\mid S\in\bar{\mathcal{X}}\},\ \mathcal{Y}_V=\{G/H'\times(S'\cap V)\mid S'\in\bar{\mathcal{Y}}\}.$$

Then, $(\mathcal{X}_U,\mathcal{Y}_V)$ is a G-Thom stratification of $f\mid G/H\times U$. Hence,

$$(\mathcal{X}=\{\bigcup_U G/H\times(S\cap U)\mid S\in\bar{\mathcal{X}}\},\ \mathcal{Y}=\{\bigcup_V G/H'\times(S'\cap V)\mid S'\in\bar{\mathcal{Y}}\})$$

is a G-Thom stratification of f.

Remark. This theorem has been announced in [3].

If X and Y have several orbit types, the situation is more complicated as the following example shows:

Example. Let $\rho_i: \boldsymbol{Z}_2\longrightarrow GL(2,\boldsymbol{R})$ $(i=1,2)$ be group representations defined by

$$\rho_1(-1)=\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix} \quad\text{and}\quad \rho_2(-1)=\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}.$$

We denote by $(\boldsymbol{R}^2,\rho_1)$ and $A=(\boldsymbol{R}^2,\rho_2)$ the $\boldsymbol{Z}_2$-spaces given by the above representations. Now, we consider the map germ $f:(R,0)\longrightarrow(A,0)$ defined by $f(x,y)=(x,xy)$. By Malgrange's preparation theorem, it is easy to show that f is an infinitesimally stable $\boldsymbol{Z}_2$-map germ. But, f is the famous example which cannot have Thom stratifications. If we get an unfolding of f as follows, we have same examples between higher dimensional $\boldsymbol{Z}_2$-spaces:

$$F:(R\times\boldsymbol{R}^s,0)\longrightarrow(A\times\boldsymbol{R}^s,0)$$

by $F(x,y,u_1,\cdots,u_s)=(f(x,y),u_1,\cdots,u_s)$, where $\boldsymbol{Z}_2$ acts trivially on $\boldsymbol{R}^s$.

This example indicates that we cannot have the "generic" set of (usual or direct analogous) Thom mappings in $C^\infty_{Z_2}(R\times\boldsymbol{R}^s,A\times\boldsymbol{R}^s)$.

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