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A CONSTRUCTION OF SPACES WITH GENERAL CONNECTIONS WHICH HAVE POINTS SWALLOWING GEODESICS

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Introduction. The purpose of this paper is to show some way by which we can construct examples of spaces admitting the existence of points which have neighborhoods such that if any one going on along a geodesic enters these neighborhoods, he will be finally swallowed at these points. It is well known that any space with a classical affine connection does not admit the existence of points as mentioned above. We mean here geometrical spaces as differentiable manifolds with general connections introduced by the present author in 1958 [1], which include affine connections as special ones. The property of these neighborhoods will be illustrated by the following typical example: Let E^2 be the Euclidean plane. Any straight line diverges to the point at infinity. Considering E^2 as the complex z-plane and bringing the point at infinity to the origin by the inversion $w=1/\bar{z}$, then straight lines of E^2 are represented by circles or straight lines through the origin $w=0$. The point $w=0$ does not belong to E^2 originally. If we annex $w=0$ to E^2, then any circular neighborhood of the origin $w=0$ has the property mentioned above. Here, it must be noticed that the flat affine connection of E^2 loses its meaning at $w=0$, since its line element is

$$ds^2=dzd\bar{z}=\frac{1}{|w|^4}dwd\bar{w}.$$

1. General connections and geodesics. Let M^n be a smooth manifold of dimension n and Γ be a general connection on M^n, which is a cross-section of the tensor product bundle $T(M^n)\otimes D^2(M^n)$ of the tangent bundle of order 1 $T(M^n)$ and the co-tangent bundle of order 2 $D^2(M^n)$ of M^n and is represented in local coordinates u^i of M^n as

$$\Gamma=\frac{\partial}{\partial u^i}\otimes(P^i_jd^2u^j+\Gamma^i_{jk}du^j\otimes du^k) \tag{1.1}$$

with local components (P^i_j, Γ^i_{jk}). If $(\bar{P}^i_j, \bar{\Gamma}^i_{jk})$ are the local components in another local coordinate system $(\bar{u}^i)$, then we obtain

(1.2) $$\bar{P}^i_j=\frac{\partial\bar{u}^i}{\partial u^k}P^k_h\frac{\partial u^h}{\partial\bar{u}^j},$$

(1.3) $$\bar{\Gamma}^i_{jk}=\frac{\partial\bar{u}^i}{\partial u^h}\left(P^h_l\frac{\partial^2 u^l}{\partial\bar{u}^k\partial\bar{u}^j}+\Gamma^h_{lm}\frac{\partial u^l}{\partial\bar{u}^j}\frac{\partial u^m}{\partial\bar{u}^k}\right),$$

since we have the rule on the differential of order 2 by definition as

$$d^2\bar{u}^i=\frac{\partial\bar{u}^i}{\partial u^j}d^2u^j+\frac{\partial^2\bar{u}^i}{\partial u^j\partial u^k}du^j\otimes du^k.$$

We see easily from (1.2) and (1.3) that $P=\frac{\partial}{\partial u^i}\otimes P^i_j du^j$ is a tensor field of type (1, 1) and Γ is a classical affine connection if $P=I$ (the Kronecker's δ).

Now, for any tangent vector field $X=\frac{\partial}{\partial u^i}X^i$ on M^n its covariant differential DX with respect to Γ is given by

$$DX=X^i{}_{,j}\frac{\partial}{\partial u^i}\otimes du^j,$$

where

(1.4) $$X^i{}_{,j}=P^i_k\frac{\partial X^k}{\partial u^j}+\Gamma^i_{kj}X^k.$$

For a given vector field $Y(t)=Y^i(t)\frac{\partial}{\partial u^i}$ along a curve $\gamma(t)$: $u^i=u^i(t)$, $a<t<b$, its covariant derivative DY/dt with respect to Γ is given by

$$\frac{DY}{dt}=\frac{DY^i}{dt}\frac{\partial}{\partial u^i},$$

where

(1.5) $$\frac{DY^i}{dt}=P^i_j\frac{dY^j}{dt}+\Gamma^i_{jk}Y^j\frac{du^k}{dt}$$

For any tensor field $Q=\frac{\partial}{\partial u^i}\otimes Q^i_j du^j$ of M^n we can define two general connections $Q\Gamma$ and ΓQ derived from Γ with local components in u^i as follows:

(1.6) $$\begin{cases} Q\Gamma=(Q^i_l P^l_j,\ Q^i_l\Gamma^l_{jk}), \\ \Gamma Q=(P^i_l Q^l_j,\ P^i_l\frac{\partial Q^l_j}{\partial u^k}+\Gamma^i_{lk}Q^l_j) \end{cases}$$

([7]). From (1.6) we see that if $P=\lambda(\Gamma)$ is isomorphic on each tangent space of M^n, then $P^{-1}\Gamma$ and ΓP^{-1} become classical affine connections.

A curve $\gamma(t)$ is called a geodesic with respect to Γ if it satisfies the equation:

(1.7) $$\frac{D}{dt}\left(\frac{du^i}{dt}\right)=\phi P^i_j\frac{du^j}{dt}$$

for a suitable function ψ of t. The parameter s for a geodesic such that

$$\frac{D}{ds}\left(\frac{du^i}{ds}\right)=0 \tag{1.8}$$

is called an affine parameter of the geodesic. We can easily see that for a geodesic $\gamma(t)$ satisfying

$$P^i_j\frac{du^j}{dt}\neq 0$$

at any point, its affine parameter is determined uniquely except affine transformations as in the case of classical affine connections.

From (1.5), (1.6) and (1.7) we obtain immediately

Lemma 1. *Let γ be a geodesic with respect to a general connection Γ of M^n, then γ is also a geodesic with respect to the general connection $Q\Gamma$, where Q is any tensor field of type* (1, 1) *on M^n, and the affine parameters of γ with respect to Γ become so for $Q\Gamma$.*

2. Special cases in which appear points swallowing geodesics. Let Γ be a general connection defined on a neighborhood of the origin in R^n with the canonical coordinates u^i and (P^i_j, Γ^i_{jk}) the components of Γ.

Now, suppose that

$$P^i_j=\rho\delta^i_j \tag{2.1}$$

and Γ^i_{jk} the Christoffel symbols made by a symmetric tensor $g_{ij}=\sigma\delta_{ij}$ $(\sigma\neq 0)$ as

$$\begin{aligned}\Gamma^i_{jk}&=\frac{1}{2}g^{il}\left(\frac{\partial g_{lk}}{\partial u^j}+\frac{\partial g_{jl}}{\partial u^k}-\frac{\partial g_{jk}}{\partial u^l}\right)\\&=\frac{1}{2\sigma}(\delta^i_j\sigma_k+\delta^i_k\sigma_j-\delta_{jk}\sigma_i),\end{aligned} \tag{2.2}$$

where $\sigma_i=\dfrac{\partial\sigma}{\partial u^i}$. Then the equation (1.8) of a geodesic with respect to Γ becomes

$$\rho\frac{d^2u^i}{ds^2}+\frac{1}{\sigma}\left(\frac{du^i}{ds}\sigma_k\frac{du^k}{ds}-\frac{1}{2}\sigma_i\sum_k\frac{du^k}{ds}\frac{du^k}{ds}\right)=0.$$

Now, regarding u^i as Euclidean coordinates we denote its inner product by $\langle\ ,\ \rangle$. Then the above equation can be expressed as

$$\rho\frac{d^2u}{ds^2}+\frac{1}{\sigma}\left(\frac{du}{ds}\langle\nabla\sigma, \frac{du}{ds}\rangle-\frac{1}{2}\nabla\sigma\langle\frac{du}{ds}, \frac{du}{ds}\rangle\right)=0, \tag{2.3}$$

where $\nabla\sigma$ is the gradient vector field of σ with respect to the Euclidean connection of R^n.

Furthermore we put the conditions for ρ and σ as

$$\rho=\rho(r) \quad \text{and} \quad \sigma=\sigma(r) \tag{2.4}$$

where

$$\langle u, u\rangle = r^2 \quad (r \geqq 0).$$

and

$$\rho(r)\neq 0 \quad \text{for } r>0. \tag{2.5}$$

Then, since we have

$$\nabla\sigma=\sigma'(r)\frac{u}{r} \quad \text{for } u\neq 0,$$

(2.3) becomes

$$\frac{d^2u}{ds^2}+\frac{\sigma'}{\rho\sigma}\frac{dr}{ds}\frac{du}{ds}-\frac{\sigma'}{2\rho\sigma r}\left|\frac{du}{ds}\right| u=0, \tag{2.6}$$

from which we see that any geodesic lies on a Euclidean plane of R^n through the origin. Therefore, we may consider (2.6) as the equation on R^2. We represent (2.6) by the polar coordinates (r, θ) of R^2. We have easily

$$u=r\begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix}, \quad \frac{du}{ds}=\frac{dr}{ds}\begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix}+r\frac{d\theta}{ds}\begin{pmatrix}-\sin\theta\\ \cos\theta\end{pmatrix},$$

$$\frac{d^2u}{ds^2}=\left\{\frac{d^2r}{ds^2}-r\left(\frac{d\theta}{ds}\right)^2\right\}\begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix}+\left\{r\frac{d^2\theta}{ds^2}+2\frac{dr}{ds}\frac{d\theta}{ds}\right\}\begin{pmatrix}-\sin\theta\\ \cos\theta\end{pmatrix}$$

and

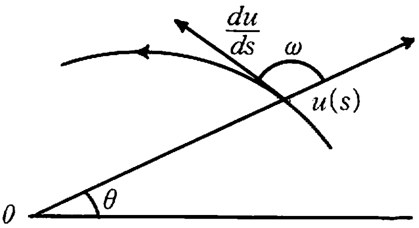
$$\left|\frac{du}{ds}\right|^2=\left(\frac{dr}{ds}\right)^2+r^2\left(\frac{d\theta}{ds}\right)^2.$$

Substituting these into (2.6), we obtain

$$\begin{cases}\dfrac{d^2r}{ds^2}+\dfrac{\sigma'}{2\rho\sigma}\left(\dfrac{dr}{ds}\right)^2-r\left(1+\dfrac{r\sigma'}{2\rho\sigma}\right)\left(\dfrac{d\theta}{ds}\right)^2=0,\\[2mm] r\dfrac{d^2\theta}{ds^2}+2\left(1+\dfrac{r\sigma'}{2\rho\sigma}\right)\dfrac{dr}{ds}\dfrac{d\theta}{ds}=0.\end{cases} \tag{2.7}$$

In order to see the behavior of a geodesic near the origin, we consider the angle ω between the position vector $u(s)$ and its tangent vector. Then we have easily

$$
(2.8)\quad \begin{cases} \left|\dfrac{du}{ds}\right| \cos\omega = \dfrac{dr}{ds}, \\ \left|\dfrac{du}{ds}\right| \sin\omega = r\dfrac{d\theta}{ds}, \\ \dfrac{d\theta}{ds} = \dfrac{1}{r}\tan\omega\ \dfrac{dr}{ds}. \end{cases}
$$



Differentiating the first equality of (2.8) and using (2.7) and (2.8), we have

$$
\begin{aligned}
\left|\frac{du}{ds}\right| \sin\omega\ \frac{d\omega}{ds} &= -\frac{d^2r}{ds^2} + \cos\omega\ \frac{d}{ds}\left|\frac{du}{ds}\right| \\
&= -\frac{d^2r}{ds^2} + \frac{\cos\omega}{2\left|\dfrac{du}{ds}\right|}\frac{d}{ds}\left\{\left(\frac{dr}{ds}\right)^2 + r^2\left(\frac{d\theta}{ds}\right)^2\right\} \\
&= -\frac{d^2r}{ds^2} + \frac{\cos^2\omega}{\dfrac{dr}{ds}}\left\{\frac{dr}{ds}\frac{d^2r}{ds^2} + r\frac{dr}{ds}\left(\frac{d\theta}{ds}\right)^2 + r^2\frac{d\theta}{ds}\frac{d^2\theta}{ds^2}\right\} \\
&= -\sin^2\omega\ \frac{d^2r}{ds^2} + r\cos^2\omega\ \left(\frac{d\theta}{ds}\right)^2 + \frac{r^2\cos^2\omega}{\dfrac{dr}{ds}}\frac{d\theta}{ds}\frac{d^2\theta}{ds^2} \\
&= \sin^2\omega\ \left\{\frac{\sigma'}{2\rho\sigma}\left(\frac{dr}{ds}\right)^2 - r\left(1+\frac{r\sigma'}{2\rho\sigma}\right)\left(\frac{d\theta}{ds}\right)^2\right\} + r\cos^2\omega\left(\frac{d\theta}{ds}\right)^2 \\
&\qquad - \frac{2r\cos^2\omega}{\dfrac{dr}{ds}}\left(1+\frac{r\sigma'}{2\rho\sigma}\right)\frac{dr}{ds}\left(\frac{d\theta}{ds}\right)^2 \\
&= \sin^2\omega\ \frac{\sigma'}{2\rho\sigma}\left(\frac{dr}{ds}\right)^2 \\
&\qquad + r\left\{-\sin^2\omega\left(1+\frac{r\sigma'}{2\rho\sigma}\right) + \cos^2\omega - 2\cos^2\omega\left(1+\frac{r\sigma'}{2\rho\sigma}\right)\right\}\left(\frac{d\theta}{ds}\right)^2 \\
&= \left\{\sin^2\omega\ \frac{\sigma'}{2\rho\sigma} + \frac{1}{r}\tan^2\omega\left(-1-(\sin^2\omega+2\cos^2\omega)\frac{r\sigma'}{2\rho\sigma}\right)\right\}\left(\frac{dr}{ds}\right)^2 \\
&= -\frac{1}{r}\tan^2\omega\ \left(1+\frac{r\sigma'}{2\rho\sigma}\right)\left(\frac{dr}{ds}\right)^2,
\end{aligned}
$$

hence

$$
\begin{aligned}
\frac{d\omega}{ds} &= -\frac{\tan^2\omega}{\left|\dfrac{du}{ds}\right| r\sin\omega}\left(1+\frac{r\sigma'}{2\rho\sigma}\right)\left(\frac{dr}{ds}\right)^2 \\
&= -\frac{\sin\omega}{\left|\dfrac{du}{ds}\right| r}\left(1+\frac{r\sigma'}{2\rho\sigma}\right)\left(\frac{1}{\cos\omega}\frac{dr}{ds}\right)^2 = -\frac{1}{r}\left(1+\frac{r\sigma'}{2\rho\sigma}\right)\left|\frac{du}{ds}\right|\sin\omega,
\end{aligned}
$$

i.e.

$$(2.9)\qquad \frac{d\omega}{ds}=-\frac{1}{r}\left(1+\frac{r\sigma'}{2\rho\sigma}\right)\left|\frac{du}{ds}\right|\sin\omega=-\left(1+\frac{r\sigma'}{2\rho\sigma}\right)\frac{d\theta}{ds}.$$

Remark. Consider E^2 near the point at infinity which is represented as $w=0$ in the complex w-plane and let its line element be

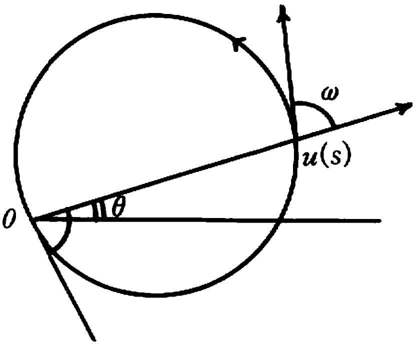
$$ds^2=\frac{1}{|w|^4}dw\,d\bar{w},\quad w=u^1+iu^2.$$

Then, we can put

$$\sigma=\frac{1}{r^4},\quad \rho=1$$

for E^2, even though the origin is outside of E^2. (2.7) and (2.9) become

$$\begin{cases}\dfrac{d^2r}{ds^2}-\dfrac{2}{r}\left(\dfrac{dr}{ds}\right)^2+r\left(\dfrac{d\theta}{ds}\right)^2=0,\\ \dfrac{d^2\theta}{ds^2}-\dfrac{2}{r}\dfrac{dr}{ds}\dfrac{d\theta}{ds}=0,\\ \dfrac{d\omega}{ds}=\dfrac{d\theta}{ds}.\end{cases}$$



Now, we suppose the following condition for $\rho(r)$ and $\sigma(r)$ as

$$(2.10)\qquad 1+\frac{r\sigma'}{2\rho\sigma}<0\quad\text{for}\quad 0\leqq r\leqq r_0$$

for a positive constant r_0.

If a geodesic enters in the disk $r\leqq r_0$ at a point (r_0,θ_0), we may put for $\omega=\omega_0$ at this point

$$(2.11)\qquad \frac{\pi}{2}\leqq\omega_0\leqq\pi$$

and put $s=s_0$ at this point in the following.

If $\omega_0=\pi$, then from (2.8) and (2.9) we have at this point

$$\frac{d\theta}{ds}=\frac{d\omega}{ds}=0$$

and we can obtain from (2.7) the equalities

$$\theta\equiv\theta_0$$

and

$$\frac{d^2 r}{ds^2}+\frac{\sigma'(r)}{2\rho(r)\sigma(r)}\left(\frac{dr}{ds}\right)^2=0,$$

which implies the equality:

$$\int_r^{r_0} e^{-\int_t^{r_0}\frac{\sigma'(y)}{2\rho(y)\sigma(y)}dy}\,dt=k_0(s-s_0) \tag{2.12}$$

where $k_0=-\frac{dr}{ds}|_{s=s_0}$. In the w-plane, the trace of this geodesic is a straight line through the origin $w=0$.

Next, if $\frac{\pi}{2}<\omega_0<\pi$, then we see from (2.9) that ω is increasing for $s \geqq s_0$ and by the argument above it follows

$$\frac{\pi}{2}<\omega<\pi \quad \text{and} \quad \frac{dr}{ds}<0. \tag{2.13}$$

Now, suppose that this geodesic is defined maximally for $s<s_1$ $(\leqq +\infty)$ and a point (r_1, θ_1) is a cluster point of the geodesic for $s \to s_1$. If $r_1>0$, this geodesic is also a geodesic with respect to the affine connection $P^{-1}\Gamma$ and s is also its affine parameter by Lemma 1. This contradicts the well-known fact for affine connections. Thus we see that

$$r_1=0$$

which shows that this geodesic must be swallowed by the origin.

Finally, we show that we can give $\rho(r)$ and $\sigma(r)$ so that they satisfy (2.10). First we take a smooth function $\sigma(t)$ as

$$\sigma(t)=\begin{cases}2 & t \leqq 0,\\ 1 & t \geqq 2r_0\end{cases} \tag{2.14}$$

and $\sigma(t)$ is decreasing for $0<t<2r_0$ for example if we take

$$\sigma(t)=\frac{e^{-1/t}+2e^{-1/(2r_0-t)}}{e^{-1/t}+e^{-1/(2r_0-t)}} \quad \text{for} \quad 0<t<2r_0,$$

$$\sigma'(t)=-\frac{\left(\frac{1}{t^2}+\frac{1}{(2r_0-t)^2}\right)e^{-\frac{1}{t}-\frac{1}{2r_0-t}}}{(e^{-1/t}+e^{-1/(2r_0-t)})^2}<0$$

Second, we determine $\rho(r)$ by the equality

$$1+\frac{r\sigma'}{2\rho\sigma}=-1,$$

i.e.

$$\rho(r)=-\frac{r\sigma'(r)}{4\sigma(r)}. \tag{2.15}$$

Then, we have

$$\rho(0)=0, \ \rho(r)>0 \quad \text{for} \quad 0<r\leqq r_0. \tag{2.16}$$

3. A way to construct spaces for any M^n which have points swallowing geodesics. In this section we show a way by which we can construct certain spaces with suitable general connections for any given manifolds which admit the points swallowing geodesics of the spaces as treated in §2.

First of all, we prepare an auxiliary funcfion of one variable. Take a positive constant r_0, we define $\psi_{r_0}(t)$ by

$$\psi_{r_0}(t)=\begin{cases} 1 & \text{for} \quad t\leqq r_0, \\ e^{-1/(2r_0-t)}/\{e^{-1/(t-r_0)}+e^{-1/(2r_0-t)}\} & \text{for} \quad r_0\leqq t\leqq 2r_0, \\ 0 & \text{for} \quad t\geqq 2r_0. \end{cases} \tag{3.1}$$

Let M^n be any n-dimensional smooth manifold and g be any Riemannian metric on M^n. Let Γ_g be the Riemannian connection on M^n determined by g. Now, take a fixed point p_0 of M^n and let $\{u^1, \cdots, u^n\}$ be a geodesic normal coordinate system with p_0 as its center defined on a neighborhood U of p_0. Setting

$$r=\sqrt{\sum u^i u^i} \qquad \text{on} \quad U,$$

we suppose that the points such that $r\leqq 2r_0$ are included in U. Now, we change the Riemannian metric g to another one $\hat{g}$ as follows. Setting

$$g=g_{ij}du^i\otimes du^j \ \text{and} \ \hat{g}=\hat{g}_{ij}du^i\otimes du^j \ \text{on} \ U,$$

we put

$$\hat{g}_{ij}=(1-\psi_{r_0}(r))g_{ij}+\psi_{r_0}(r)\sigma(r)\delta_{ij} \quad \text{on} \ U, \tag{3.2}$$

where $\sigma(t)$ is the auxiliary function defined by (2.14) and $\hat{g}=g$ outside of U. By (3.1) and (3.2), we see

$$\hat{g}_{ij}=\sigma(r)\delta_{ij} \quad \text{for} \quad r\leqq r_0$$

and

$$\hat{g}_{ij}=g_{ij} \quad \text{for} \quad r\geqq 2r_0.$$

Let $\hat{\Gamma}=\Gamma_{\hat{g}}$ be the Riemannian connection of M^n determined by $\hat{g}$.

Next, we consider $\hat{\Gamma}$ as a general connection on M^n and construct another general connection Γ as follows.

We put

$$\Gamma=\hat{\Gamma} \qquad \text{outside of} \ U,$$

and, denoting them by the local components with respect to the coordinates $u^1, \cdots, u^n$ as

$$\Gamma=(P^i_j,\ \Gamma^i_{jk}) \quad \text{and} \quad \bar{\Gamma}=(\delta^i_j,\ \bar{\Gamma}^i_{jk})$$

inside of U, we put

$$\text{(3.3)} \qquad \begin{cases} P^i_j=\{\psi_{r_0}(r)\rho(r)+(1-\psi_{r_0}(r))\}\delta^i_j, \\ \Gamma^i_{jk}=\bar{\Gamma}^i_{jk}, \end{cases}$$

where $\rho(r)$ is the function defined by (2.15) with $\sigma(r)$ by (2.14). It is clear that we have also $\Gamma=\bar{\Gamma}$ for $r \geqq 2r_0$ in U and $\Gamma=(\rho\delta^i_j,\ \bar{\Gamma}^i_{jk})$ for $r \leqq r_0$. Thus, we see that if any geodesic with respect to the general connection Γ enters into the neighborhood

$$r(u) < r_0$$

then it must be swallowed finally by the point p_0 by means of the argument in §2.

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