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SOME POLYNOMIAL IDENTITIES AND COMMUTATIVITY OF s-UNITAL RINGS. II

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Throughout the present paper, R will represent a ring with center C. Let Q be the set of all quasi-regular elements in R, and N the set of all nilpotent elements in R. Let D be the commutator ideal of R. Given $x, y \in R$, we let $[x,y]=xy-yx$ and $x \circ y=x+y-xy$. Let n be a (fixed) positive integer, and consider the following ring-properties:

A : R is commutative.
B_0 : For any $x, y \in R$, $(x-xy) \circ (y-yx)=0$ if and only if $x=y$.
B_1 : $(x-x^2) \circ (x-x^2)=0$ for all $x \in R$.
B_2 : $(x-x^2)^3=0$ for all $x \in R$.
B_3 : $x-x^2 \in Q$ for all $x \in R$.
B_4 : $x \circ x=0$ for all $x \in Q$.
$P_3'(n+1)$: $[(xy)^{n+1}-(yx)^{n+1},x]=0$ for all $x, y \in R$.
$P_{10}(n)$: $[x^n,y^n]=0$ for all $x, y \in R$.
$P_{11}(n+1)$: $(xy)^{n+1}=x^{n+1}y^{n+1}$ for all $x, y \in R$.
$P_{11}''(n+1)$: $[[(xy)^{n+1}-x^{n+1}y^{n+1},x],z]=0=[[(xy)^{n+1}-x^{n+1}y^{n+1},y],z]$ for all $x, y, z \in R$.
$P_{12}(n+1)$: $[x-x^{n+1},y-y^{n+1}]=0$ for all $x, y \in R$.
$P_{12}'(n)$: $[[x-x^n,y-y^n],y]=0$ for all $x, y \in R$.
$Q(n)$: For any $x, y \in R$, $n[x,y]=0$ implies $[x,y]=0$.

The present objective is to prove the following theorems.

Theorem 1. *If R is an s-unital ring, then $B_1 \Leftrightarrow B_2 \wedge B_4 \Leftrightarrow B_3 \wedge B_4$.*

Theorem 2. *If R is an s-unital ring, then $B_0 \Leftrightarrow B_2 \wedge B_4 \wedge A \Leftrightarrow B_3 \wedge B_4 \wedge A$.*

Theorem 3. *If R is an s-unital ring, then $P_{10}(n) \wedge P_{11}''(n+1) \wedge Q(n) \Leftrightarrow P_{10}(n) \wedge P_3'(n+1) \wedge Q(n) \Leftrightarrow P_{10}(n) \wedge P_{12}'(n) \wedge Q(n) \Leftrightarrow A$.*

Theorem 4. *If R is an s-unital ring, then $P_{11}(n+1) \wedge P_{12}(n+1) \wedge Q(n) \Leftrightarrow A$. If furthermore n is odd, then $P_{11}(n+1) \wedge P_{12}(n+1) \Leftrightarrow A$.*

Obviously, B_1, $P_3'(n+1)-P_{12}'(n)$ and $Q(n)$ are H-properties, A, B_2 and

B_4 are F-properties (in the sense of [4]), $B_2 \Rightarrow B_3$, and $B_2 \wedge B_4 \Rightarrow B_1$. Furthermore, in case R has 1, B_1 becomes

$$B_1^* : (1-x+x^2)^2=1 \text{ for all } x \in R.$$

Proof of Theorem 1. According to [4, Proposition 1] and the facts just mentioned above, it is enough to show that if R has 1 then B_1^* implies B_2 and B_4.

Setting $x=2$ in B_1^*, we find $8=0$. Replacing x by $-x$ in B_1^* to get

$$(1) \qquad (1+x+x^2)^2=1 \text{ for all } x \in R.$$

From B_1^* and (1), we get $4x(1+x^2)=0$, i.e.,

$$(2) \qquad 4x=4x^3.$$

Replacing x by $1+x$ in (2) and noting that $8=0$, we get $4(1+x)=4(1+x)^3 =4+4x+4x^2-4x^3$. Combining this with (2), we see that

$$(3) \qquad 4x=4x^2.$$

Now, as $(x-x^2)^2=2(x-x^2)$ by B_1^*, (3) shows that $(x-x^2)^3=2(x-x^2)^2= 4(x-x^2)=0$, which proves B_2. From this we readily see that

$$(4) \qquad x^3=0 \text{ for all } x \in Q.$$

Hence, for all $x \in Q$, we see that $(1-x)^{-1}=1+x+x^2$. Thus, $(1-x)^2= (1+x+x^2)^{-2}=1$ by (1), and R has the property B_4.

Proof of Theorem 2. It is easy to see that $B_3 \wedge B_4 \wedge A$ implies B_0. Since B_0 implies B_1, and therefore $B_2 \wedge B_4$ by Theorem 1, it remains only to prove that B_0 implies A.

According to B_2, $x^3=0$ for all $x \in Q$. Furthermore, since $(E_{11}+E_{12}+E_{21})-(E_{11}+E_{12}+E_{21})^2=-1$ in $(\mathrm{GF}(p))_2$ (p a prime), D is a nil ideal by [4, Proposition 2], and hence $Q=N$ is an ideal. By B_4, for all $x, y \in Q$,

$$x \circ y=x \circ (x \circ y \circ x \circ y) \circ y=y \circ x,$$

which shows that Q is a commutative ideal. In order to show A, choose arbitrary $a\in Q$ and $b\in R$, and set $x=(1+a)b$ and $y=b(1+a)$. Obviously, $x-y=[a,b]$. Since Q is a commutative ideal, by B_2 we see that $b^2a+ab^2-[a,b]=2b^2a+[a,b^2-b]=2b^2a$, and $[b,ba]$ $(=[b^2,ba])=[-b,ba]$, i.e., $[2b,ba]=0$. Hence,

$$\begin{aligned}[x,y]-(x-y)&=[x,x-[a,b]]-[a,b]=[[a,b],(1+a)b]-[a,b]\\ &=[[a,b],b]-[a,b]=-2bab+b^2a+ab^2-[a,b]\\ &=2b^2a-2bab=[2b,ba]=0,\end{aligned}$$

whence it follows that

$$(5) \qquad xy-x=yx-y.$$

Now, by B_2 we have

$$xy-x=(1+a)b^2(1+a)-(1+a)b=(b^2-b)+(ab^2+b^2a+ab^2a-ab)\in Q.$$

Accordingly, by (5) and B_4, $(x-xy)\circ(y-yx)=0$. Thus, by B_0, it follows that $x=y$, namely $ab=ba$. This means that Q is contained in C. Thus, by B_2 and [1, Theorem 19], we see that R is commutative.

Remark. (1) Theorem 2 provides a complete solution to Problem E 2825 suggested by A. Melter [Amer. Math. Monthly 87 (1980), 220].

(2) As was claimed in the proof of Theorem 2, if a ring R with Jacobson radical J has the property B_2 then $x^3=0$ for all $x\in J$ and R/J is a Boolean ring. It is obvious that the converse is also true.

(3) Let R be a ring with 1 having the property B_0. If $Q^2=0$, then R is a trivial extension of the Boolean ring R/Q by Q. Conversely, if B is a Boolean ring with 1 and M is a B-module, then the trivial extension of B by M has the property B_0.

(4) Let $S=\boldsymbol{Z}/8\boldsymbol{Z}$, and $R=\{\begin{pmatrix} s & t\\ 0 & s\end{pmatrix}\mid s\in S,\ t\in 4s\}$. Then R is a commutative ring with 1 and $Q=\{\begin{pmatrix} s & t\\ 0 & s\end{pmatrix}\mid s\in 2S,\ t\in 4S\}$. Since $\begin{pmatrix} 2 & 4\\ 0 & 2\end{pmatrix}^2\neq 0$, Q contains an element of (nilpotency) index 3. It is easy to see that R has the property $B_2\wedge B_4$, and therefore B_0 .

In preparation for proving Theorems 3 and 4, we state the next :

Lemma 1. *Let R be a ring with* 1.

(i) *If R has the property $P_{10}(n)\wedge Q(n)$, then $[a,x^n]=0$ for all $a\in N$ and $x\in R$, and N is a commutative ideal containing D.*

(ii) *If R has the property $P_{10}(n)\wedge Q(n)$ and $[[[a,x],x],x]=0$ for all $a\in N$ and $x\in R$, then R is commutative.*

(iii) *If r is an element of R such that $r^2x=rxr=xr^2$ for all $x\in R$ then*

$$\{(r+1)^{n+1}x^{n+1}-((r+1)x)^{n+1}\}-\{x^{n+1}(r+1)^{n+1}-(x(r+1))^{n+1}\}=n[r,x^{n+1}].$$

Proof. (i) is included in [3, Lemma 2], and the proof of (iii) is straightforward.

(ii) Since $[[a,x],x^n]=0$ by (i), we have $n[[a,x],x]x^{n-1}=0$. Replacing

x by $x+1$ in this last equation, we get $n[[a,x],x](x+1)^{n-1}=0$. The last two identities are easily seen to imply $n[[a,x],x]=0$, and hence by $Q(n)$, $[[a,x],x]=0$. Arguing in a similar manner, we see that $0=[a,x^n]=n[a,x]x^{n-1}$, and hence (as argued above) $[a,x]=0$. This proves that $N \subseteq C$, and thus in particular, $D \subseteq C$ by (i). Now, it is easy to see that R is commutative, using $P_{10}(n)$, $Q(n)$, and the above argument of replacing first x by $x+1$ and then y by $y+1$.

Proof of Theorem 3. We prove that, under the hypothesis $P_{10}(n) \wedge Q(n)$, each of $P''_{11}(n+1)$, $P'_3(n+1)$ and $P'_{12}(n)$ implies A. According to [4, Proposition 1], we may assume that R has 1. At any rate, by Lemma 1 (i), $[a,x^n]=0$ for all $a \in N$ and $x \in R$, and N is a commutative ideal containing D, and therefore $N^2 \subseteq C$. By Lemma 1 (ii), it suffices to show that $[[[a,x],x],x]=0$ for all $a \in N$ and $x \in R$. First, suppose $P''_{11}(n+1)$. Then, by Lemma 1 (iii), for any $a \in N$ and $x \in R$ we have $[n[a,x^{n+1}],x] \in C$. Hence, by $Q(n)$ and $[a,x^n]=0$, it follows that

$$0=[[[a,x^{n+1}],x],x]=[[[a,x]x^n,x],x]=[[[a,x],x],x]x^n,$$

which implies $[[[a,x],x],x]=0$. Next, suppose $P'_3(n+1)$. Then, noting that $[x^n,a]=0$ $(a \in N,\ x \in R)$, we can easily see that

$$0=[\{x(a+1)\}^{n+1}-\{(a+1)x\}^{n+1},x]=[[x^{n+1},a],x]=[[x,a],x]x^n,$$

and hence $[[a,x],x]=0$. Finally, suppose $P'_{12}(n)$. Then, it can be seen that $[[a,x],x]=[[a,x-x^n],x]=0$ for all $a \in N$ and $x \in R$.

Proof of Theorem 4. We prove that $P_{11}(n+1) \wedge P_{12}(n+1) \wedge Q(n) \Rightarrow A$. According to [4, Proposition 1], we may assume that R has 1. By Chacron's theorem [2, Theorem 1], N is a commutative ideal containing D. Thus, as in the proof of Theorem 3, we can show that $[a,y^{n+1}]=0$ for all $a \in N$ and $y \in R$, now. Combining this with $P_{12}(n+1)$, we readily obtain $[a,y]=0$, that is $N \subseteq C$.

Now, let x, y be arbitrary elements of R. Then

$$\begin{aligned} 0&=[xy-(xy)^{n+1},y-y^{n+1}]=[xy-x^{n+1}y^{n+1},y-y^{n+1}] \\ &=[x,y-y^{n+1}]y-[x^{n+1},y-y^{n+1}]y^{n+1}=[x,y-y^{n+1}](y-y^{n+1}). \end{aligned}$$

Hence, $x(y-y^{n+1})^2=(y-y^{n+1})x(y-y^{n+1})=(y-y^{n+1})^2x$. In view of Lemma 1 (iii), this enables us to see that $n[y-y^{n+1},x^{n+1}]=0$, whence it follows that $[y-y^{n+1},x^{n+1}]=0$. Combining this with $[y-y^{n+1},x-x^{n+1}]=0$, we obtain $[y-y^{n+1},x]=0$. Thus, by [1, Theorem 19], R is commutative.

Henceforth, suppose that n is odd. Replace y by $-y$ in $P_{12}(n+1)$ and subtract the result from $P_{12}(n+1)$ to get $[x-x^{n+1},2y]=0$. Repeat this process for x in $[x-x^{n+1},2y]=0$ to get $4[x,y]=0$. Since n is odd, the last equation implies $Q(n)$.

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